

A WEAK HARNACK ESTIMATE FOR SUPERSOLUTIONS TO THE POROUS MEDIUM EQUATION

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ABSTRACT. In this work, we prove a weak Harnack estimate for the weak supersolutions to the porous medium equation. The proof is based on a priori estimates for the supersolutions and measure theoretical arguments.

1. INTRODUCTION

In this paper, we will prove a weak Harnack estimate for the weak supersolutions to the porous medium equation

$$u_t - \Delta u^m = 0 \quad \text{in } \Omega_{T_0}, \quad (1.1)$$

where $\Omega_{T_0} = \Omega \times (0, T_0)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. In this work, we consider only the degenerate case $m > 1$. In the singular case $m < 1$, a different proof is needed. For the general theory of the equation, we refer to [1], [16] and [17].

We consider the weak supersolutions, which are defined in the usual way, with test functions under the integral sign, as weak solutions to the inequality

$$u_t - \Delta u^m \geq 0.$$

Throughout the work, we assume, that the weak supersolutions are non-negative. Properties of supersolutions to the PME are considered in [10] and [11]. In the latter one, also unbounded supersolutions, which are defined via comparison to weak solutions, are treated. In the case of the evolutionary p -Laplace equation, some interesting phenomena are observed in the case of unbounded supersolutions, see [14]. However, in the theory of the porous medium equation, there is a missing link, namely the weak Harnack estimate for the supersolutions, which is the main result of this work.

Theorem 1.1. *Let $u > 0$ be a weak supersolution in $\Omega_{T_0} \supset B(x_0, 8\rho) \times (0, T_0)$. Then, there exist constants $C_1, C_2 > 0$ depending on m and n , such that for almost every $t_0 \in (0, T_0)$, the following inequality holds*

$$\int_{B(x_0, \rho)} u(x, t_0) dx \leq \left(\frac{C_1 \rho^2}{T_0 - t_0} \right)^{1/(m-1)} + C_2 \operatorname{ess\,inf}_Q u,$$

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where

$$Q = B(x_0, 4\rho) \times (t_0 + \tau/2, t_0 + \tau) \quad \text{and} \\ \tau = \min \left\{ T_0 - t_0, C_1 \rho^2 \left(\int_{B(x_0, \rho)} u(x, t_0) dx \right)^{-(m-1)} \right\}.$$

Apart from the classification of unbounded supersolutions, the Harnack estimates play a significant role in the regularity theory of partial differential equations. The parabolic Harnack estimates have attracted a lot of interest since the result of Moser in [15]. More recently, the Harnack's inequality for the weak solutions to p -Laplace type equations was proved in [5]. Later, the weak Harnack estimate for weak supersolutions to p -Laplace type equations was proved in [13]. Finally, the various Harnack estimates for nonlinear parabolic equations are collected in [6], where also the result for weak supersolutions to the PME is presented.

Even though the result is probably known to experts, it seems to be difficult to find a reference with a complete proof. The purpose of this work is to present a proof for the weak Harnack estimate in full detail. As pointed out in [6], the structure of the proof is similar to the one of p -Laplace type equations. However, there are numerous issues to be taken care of. For instance, constants cannot be added to solutions. Moreover, in the energy estimates, usually we can only control the norm of ∇u^m , which raises some technical challenges in the arguments.

One novelty of our proof is that we are able to bypass the assumption $u > \delta > 0$ in the Caccioppoli estimates and assume only $u > 0$. This is done by choosing a clever test function, introduced in the context of doubly nonlinear equation in [9]. Thus we are able to prove the Harnack estimate for positive supersolutions directly, without approximation by supersolutions, that are bounded away from zero.

2. PRELIMINARIES

Throughout the work, we will denote a bounded domain in \mathbb{R}^n by Ω . Moreover, we make a technical assumption $B(x_0, 8\rho) \subset \Omega$. We work with the space-time cylinders $\Omega_{T_0} = \Omega \times (0, T_0) \subset \mathbb{R}^{n+1}$. The parabolic boundary $\partial_p U$ of a space-time cylinder $U = B \times (t_1, t_2)$ is defined as $\partial_p U = B \times \{t_1\} \cup \partial B \times (t_1, t_2)$. Similarly, the conjugate parabolic boundary is defined as $\partial^p U = B \times \{t_2\} \cup \partial B \times (t_1, t_2)$.

Definition 2.1. A function $u \in L^2_{loc}(0, T; H^1_{loc}(\Omega))$ is a weak supersolution to (1.1), if $u^m \in L^2_{loc}(0, T; H^1_{loc}(\Omega))$ and u satisfies

$$\iint_{\Omega_T} (-u\varphi_t + \nabla u^m \cdot \nabla \varphi) dx dt \geq 0$$

for all test functions $\varphi \in C_0^\infty(\Omega_T)$, such that $\varphi \geq 0$.

First, we will prove some Caccioppoli type estimates for the supersolutions. In order to prove the estimates, we need to use a test function depending on the supersolution u itself. However, no regularity for u is assumed in the time variable, and thus we need to regularize the function. We will use the averaged function

$$u^*(x, t) = \frac{1}{\sigma} \int_0^t e^{\frac{s-t}{\sigma}} u(x, s) ds \quad (2.1)$$

to avoid the possibly nonexistent quantity u_t . Now, the function u^* satisfies the following inequality

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} \left((\nabla u^m)^* \cdot \nabla \varphi + \varphi \frac{\partial u^*}{\partial t} \right) dx dt \geq 0 \quad (2.2)$$

for every non-negative test function $\varphi \in L^2(0, T; H_0^1(\Omega))$. For the properties of u^* , we refer to [10].

Lemma 2.2. *Let u be a non-negative weak supersolution in Ω_T and let $\zeta \in C_0^\infty(\Omega_T)$ be a smooth cut-off function, such that $0 \leq \zeta \leq 1$. Then for any $k \in \mathbb{R}$ the following holds.*

$$\begin{aligned} & \int_0^T \int_{\Omega} u^{m-1} |\nabla(u-k)_-|^2 \zeta^2 dx dt + \operatorname{ess\,sup}_{t \in (0, T)} \int_{\Omega} \zeta^2 (u-k)_-^2 dx \\ & \leq C \left(\int_0^T \int_{\Omega} (u-k)_-^2 \zeta |\zeta_t| dx dt + \int_0^T \int_{\Omega} u^{m-1} (u-k)_-^2 |\nabla \zeta|^2 dx dt \right). \end{aligned}$$

Proof. Take $\tau_1, \tau_2 \in (0, T)$ such that $\tau_1 < \tau_2$ and let u^* denote the averaged function, defined in (2.1). Take a test function $\varphi = (u^* - k)_- \zeta^2$ in (2.2), where ζ is a smooth cut-off function, such that $0 \leq \zeta \leq 1$. Thus we have

$$\begin{aligned} 0 & \leq \int_{\tau_1}^{\tau_2} \int_{\Omega} (\nabla u^m)^* \cdot \nabla (u^* - k)_- \zeta^2 dx dt \\ & + \int_{\tau_1}^{\tau_2} \int_{\Omega} 2(u^* - k)_- \zeta (\nabla u^m)^* \cdot \nabla \zeta dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} (u^* - k)_- \zeta^2 \frac{\partial u^*}{\partial t} dx dt \\ & = I_1 + I_2 + I_3. \end{aligned}$$

We observe, that

$$\begin{aligned} I_3 & = \int_{\tau_1}^{\tau_2} \int_{\Omega} (u^* - k)_- \zeta^2 \frac{\partial u^*}{\partial t} dx dt = -\frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} \zeta^2 \frac{\partial}{\partial t} (u^* - k)_-^2 dx dt \\ & = \frac{1}{2} \int_{\Omega} \zeta(x, \tau_1)^2 (u(x, \tau_1)^* - k)_-^2 dx - \frac{1}{2} \int_{\Omega} \zeta(x, \tau_2)^2 (u(x, \tau_2)^* - k)_-^2 dx \\ & + \int_{\tau_1}^{\tau_2} \int_{\Omega} (u^* - k)_-^2 \zeta \zeta_t dx dt. \end{aligned}$$

Thus, we may let $\sigma \rightarrow 0$ to get

$$\begin{aligned} I_3 \rightarrow & \frac{1}{2} \int_{\Omega} \zeta(x, \tau_1)^2 (u(x, \tau_1) - k)_-^2 dx - \frac{1}{2} \int_{\Omega} \zeta(x, \tau_2)^2 (u(x, \tau_2) - k)_-^2 dx \\ & + \int_{\tau_1}^{\tau_2} \int_{\Omega} (u - k)_-^2 \zeta \zeta_t dx dt. \end{aligned}$$

Now, we may write I_1 as

$$\begin{aligned} I_1 \rightarrow & \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-1} \nabla u \cdot \nabla (u - k)_- \zeta^2 dx dt \\ & = - \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-1} |\nabla (u - k)_-|^2 \zeta^2 dx dt. \end{aligned}$$

Finally, we will use Young's inequality to control I_2 . We have

$$\begin{aligned} \lim_{\sigma \rightarrow 0} |I_2| & \leq 2m \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{m-1} (u - k)_- |\nabla (u - k)_-| |\zeta| |\nabla \zeta| dx dt \\ & \leq \frac{m}{2} \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{m-1} |\nabla (u - k)_-|^2 \zeta^2 dx dt \\ & \quad + 2m \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{m-1} (u - k)_-^2 |\nabla \zeta|^2 dx dt. \end{aligned}$$

Collecting the facts, we get

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{m-1} |\nabla (u - k)_-|^2 \zeta^2 dx dt + \int_{\Omega} \zeta(x, \tau_2)^2 (u(x, \tau_2) - k)_-^2 dx \\ & - \int_{\Omega} \zeta(x, \tau_1)^2 (u(x, \tau_1) - k)_-^2 dx \\ & \leq C \left(\int_{\tau_1}^{\tau_2} \int_{\Omega} (u - k)_-^2 \zeta |\zeta_t| dx dt + \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{m-1} (u - k)_-^2 |\nabla \zeta|^2 dx dt \right). \end{aligned} \tag{2.3}$$

Taking the supremum over τ_2 and letting $\tau_1 \rightarrow 0$ concludes the proof. \square

Remark 2.3. By choosing a test function $\varphi_j = \zeta \eta_j$ in (2.2), where $\zeta \in C_0^\infty(\Omega)$ and $\eta_j \in C_0^\infty(-\varepsilon, \tau + \varepsilon)$, such that $\eta_j \rightarrow \chi_{[0, \tau]}$, we may integrate by parts in the time variable and let $j \rightarrow 0$ to get the inequality

$$\int_{\Omega} u(x, \tau) \zeta(x) dx \geq \int_{\Omega} u(x, 0) \zeta(x) dx - \int_0^\tau \int_{\Omega} |\nabla u^m \cdot \nabla \zeta| dx dt. \tag{2.4}$$

Lemma 2.4. *Let u be a weak supersolution in Ω_T , such that $u > 0$ and let $\zeta \in C_0^\infty(\Omega_T)$ be a smooth cut-off function, such that $0 \leq \zeta \leq 1$. Let*

$\varepsilon > 0, \varepsilon \neq 1$. Then u satisfies

$$\begin{aligned} & \int_0^T \int_{\Omega} m u^{m-\varepsilon-2} \zeta^2 |\nabla u|^2 dx dt + \frac{1}{\varepsilon|1-\varepsilon|} \operatorname{ess\,sup}_{t \in (0,T)} \int_{\Omega} u^{1-\varepsilon} \zeta^2 dx \\ & \leq \frac{C_1 m}{\varepsilon^2} \int_0^T \int_{\Omega} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \frac{C_2}{\varepsilon|1-\varepsilon|} \int_0^T \int_{\Omega} u^{1-\varepsilon} \zeta |\zeta_t| dx dt. \end{aligned}$$

Proof. As in the proof of Lemma 2.2, take $\tau_1, \tau_2 \in (0, T)$ such that $\tau_1 < \tau_2$ and let u^* denote the averaged function, defined in (2.1).

For $\lambda > 0$, we define the dampening function

$$H_{\lambda}(s) = \begin{cases} \lambda^{-\varepsilon} + \varepsilon \lambda^{-1-\varepsilon}(\lambda - s), & \text{if } 0 \leq s \leq \lambda, \\ s^{-\varepsilon}, & \text{if } s > \lambda. \end{cases}$$

Now

$$H'_{\lambda}(s) = \begin{cases} -\varepsilon \lambda^{-1-\varepsilon}, & \text{if } 0 \leq s \leq \lambda, \\ -\varepsilon s^{-1-\varepsilon}, & \text{if } s > \lambda. \end{cases}$$

Thus $H_{\lambda}(s)$ is continuously differentiable. We denote

$$h_{\lambda}(s) = \begin{cases} \int_0^s H_{\lambda}(r) dr, & \text{if } \varepsilon < 1, \\ \int_s^{\infty} H_{\lambda}(r) dr, & \text{if } \varepsilon > 1. \end{cases}$$

We choose a test function $\varphi = H_{\lambda}(u^*)\zeta^2$ in (2.2), where $\zeta \in C_0^{\infty}(\Omega_T)$ is a smooth cut-off function, such that $0 \leq \zeta \leq 1$. In order to let $\sigma \rightarrow 0$, we have to deal with the time derivative of u^* appearing in (2.2). We integrate by parts to get

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_{\Omega} \varphi \frac{\partial u^*}{\partial t} dx dt = \int_{\tau_1}^{\tau_2} \int_{\Omega} H_{\lambda}(u^*) \frac{\partial u^*}{\partial t} \zeta^2 dx dt = \iota \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{\partial h_{\lambda}(u^*)}{\partial t} \zeta^2 dx dt \\ & = -\iota \int_{\tau_1}^{\tau_2} \int_{\Omega} h_{\lambda}(u^*) (\zeta^2)_t dx dt \\ & + \iota \int_{\Omega} h_{\lambda}(u^*(x, \tau_2)) \zeta(x, \tau_2)^2 dx - \iota \int_{\Omega} h_{\lambda}(u^*(x, \tau_1)) \zeta(x, \tau_1)^2 dx. \end{aligned}$$

Here $\iota = \operatorname{sign}(1 - \varepsilon)$. Now we may let $\sigma \rightarrow 0$ to obtain

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-1} \nabla u \cdot \nabla \varphi - \iota \int_{\tau_1}^{\tau_2} \int_{\Omega} h_{\lambda}(u) (\zeta^2)_t dx dt \\ & + \iota \int_{\Omega} h_{\lambda}(u(x, \tau_2)) \zeta(x, \tau_2)^2 dx - \iota \int_{\Omega} h_{\lambda}(u(x, \tau_1)) \zeta(x, \tau_1)^2 dx \geq 0. \end{aligned} \tag{2.5}$$

We observe

$$m u^{m-1} \nabla u \cdot \nabla \varphi = m u^{m-1} H'_{\lambda}(u) |\nabla u|^2 \zeta^2 + 2 m u^{m-1} H_{\lambda}(u) \zeta \nabla \zeta \cdot \nabla u.$$

We may estimate the second term on the right hand side by

$$\begin{aligned}
|2mu^{m-1}H_\lambda(u)\zeta\nabla\zeta\cdot\nabla u| &\leq 2mu^{m-1}H_\lambda(u)\zeta|\nabla\zeta||\nabla u| \\
&= m\left(|\nabla u|\zeta u^{\frac{m-1}{2}}|H'_\lambda(u)|^{\frac{1}{2}}\right)\left(2\frac{H_\lambda(u)}{|H'_\lambda(u)|^{\frac{1}{2}}}u^{\frac{m-1}{2}}|\nabla\zeta|\right) \\
&\leq \frac{m}{2}\left(u^{m-1}|H'_\lambda(u)|\zeta^2|\nabla u|^2 + 4\frac{H_\lambda(u)^2}{|H'_\lambda(u)|}u^{m-1}|\nabla\zeta|^2\right).
\end{aligned}$$

Thus, we have the estimate

$$\begin{aligned}
&-mu^{m-1}\nabla u\cdot\nabla\varphi \\
&\geq -mu^{m-1}H'_\lambda(u)|\nabla u|^2\zeta^2 - 2mu^{m-1}H_\lambda(u)\zeta\nabla\zeta\cdot\nabla u \\
&\geq mu^{m-1}|H'_\lambda(u)||\nabla u|^2\zeta^2 - |2mu^{m-1}H_\lambda(u)\zeta\nabla\zeta\cdot\nabla u| \\
&\geq \frac{m}{2}u^{m-1}|H'_\lambda(u)||\nabla u|^2\zeta^2 - 2m\frac{H_\lambda(u)^2}{|H'_\lambda(u)|}u^{m-1}|\nabla\zeta|^2.
\end{aligned}$$

Using this estimate in (2.5) gives

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{\Omega}mu^{m-1}|H'_\lambda(u)||\nabla u|^2\zeta^2\,dx\,dt + \iota\int_{\Omega}h_\lambda(u(x,\tau_1))\zeta(x,\tau_1)^2\,dx \\
&- \iota\int_{\Omega}h_\lambda(u(x,\tau_2))\zeta(x,\tau_2)^2\,dx \\
&\leq C\int_{\tau_1}^{\tau_2}\int_{\Omega}\left(m\frac{H_\lambda(u)^2}{|H'_\lambda(u)|}u^{m-1}|\nabla\zeta|^2 + |h_\lambda|\zeta|\zeta_t|\right)\,dx\,dt.
\end{aligned}$$

Finally, we will show, that we get the correct Caccioppoli estimate as $\lambda \rightarrow 0$. We note that $H_\lambda(s)$, $H'_\lambda(s)$ and $h_\lambda(s)$ are decreasing with respect to λ . Moreover, if $\varepsilon < 1$

$$\lim_{\lambda \rightarrow 0} h_\lambda(u) = \lim_{\lambda \rightarrow 0} \int_0^u H_\lambda(s)\,ds = \int_0^u \lim_{\lambda \rightarrow 0} H_\lambda(s)\,ds = \frac{1}{1-\varepsilon}u^{1-\varepsilon}$$

and if $\varepsilon > 1$

$$\lim_{\lambda \rightarrow 0} h_\lambda(u) = \lim_{\lambda \rightarrow 0} \int_u^\infty H_\lambda(s)\,ds = \int_u^\infty \lim_{\lambda \rightarrow 0} H_\lambda(s)\,ds = \frac{-1}{1-\varepsilon}u^{1-\varepsilon}.$$

Here we used the monotone convergence theorem. We conclude $h_\lambda(u) \rightarrow \frac{1}{|1-\varepsilon|} u^{1-\varepsilon}$ as $\lambda \rightarrow 0$. Again, by the monotone convergence theorem, we conclude

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-1} |H'_\lambda(u)| |\nabla u|^2 \zeta^2 dx dt &\rightarrow \varepsilon \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-2-\varepsilon} |\nabla u|^2 \zeta^2 dx dt, \\ \int_{\tau_1}^{\tau_2} \int_{\Omega} |h_\lambda(u)| \zeta |\zeta_t| dx dt &\rightarrow \frac{1}{|1-\varepsilon|} \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{1-\varepsilon} \zeta |\zeta_t| dx dt \quad \text{and} \\ \iota \int_{\Omega} h_\lambda(u(x, \tau_i)) \zeta(x, \tau_i)^2 dx &\rightarrow \frac{\iota}{|1-\varepsilon|} \int_{\Omega} u(x, \tau_i)^{1-\varepsilon} \zeta(x, \tau_i)^2 dx, \quad i \in \{1, 2\}. \end{aligned}$$

In order to control the term involving $\frac{H_\lambda(u)^2}{|H'_\lambda(u)|}$, we observe

$$\frac{H_\lambda(u)^2}{|H'_\lambda(u)|} = \begin{cases} \left(\frac{\lambda^{\frac{1-\varepsilon}{2}}}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} \lambda^{\frac{-1-\varepsilon}{2}} (\lambda - u) \right)^2, & \text{if } 0 \leq u \leq \lambda \\ \frac{u^{1-\varepsilon}}{\varepsilon}, & \text{if } u > \lambda. \end{cases}$$

Therefore, if $\varepsilon < 1$,

$$\frac{H_\lambda(u)^2}{|H'_\lambda(u)|} \leq \frac{H_1(u)^2}{|H'_1(u)|}$$

and if $\varepsilon > 1$,

$$\frac{H_\lambda(u)^2}{|H'_\lambda(u)|} \leq \frac{u^{1-\varepsilon}}{\varepsilon}.$$

We may assume

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt < \infty, \quad (2.6)$$

and therefore use the dominated convergence theorem to conclude

$$\int_{\tau_1}^{\tau_2} \int_{\Omega} m \frac{H_\lambda(u)^2}{|H'_\lambda(u)|} u^{m-1} |\nabla \zeta|^2 dx dt \rightarrow \frac{1}{\varepsilon} \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-\varepsilon} |\nabla \zeta|^2 dx dt.$$

We note, that if the integral in (2.6) is infinite, the estimate holds. Collecting the facts gives

$$\begin{aligned} &\varepsilon \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-2-\varepsilon} |\nabla u|^2 \zeta^2 dx dt + \frac{\iota}{|1-\varepsilon|} \int_{\Omega} u(x, \tau_1)^{1-\varepsilon} \zeta(x, \tau_1)^2 dx \\ &\quad - \frac{\iota}{|1-\varepsilon|} \int_{\Omega} u(x, \tau_2)^{1-\varepsilon} \zeta(x, \tau_2)^2 dx \\ &\leq \frac{C_1}{\varepsilon} \int_{\tau_1}^{\tau_2} \int_{\Omega} m u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \frac{C_2}{|1-\varepsilon|} \int_{\tau_1}^{\tau_2} \int_{\Omega} u^{1-\varepsilon} \zeta |\zeta_t| dx dt. \end{aligned} \quad (2.7)$$

Take $\delta > 0$. There exists $\tilde{\tau} \in (0, T)$, such that

$$\int_{\Omega} u(x, \tilde{\tau})^{1-\varepsilon} \zeta(x, \tilde{\tau})^2 dx \geq \operatorname{ess\,sup}_{t \in (0, T)} \int_{\Omega} u(x, \tilde{\tau})^{1-\varepsilon} \zeta(x, \tilde{\tau})^2 dx - \delta.$$

If $\varepsilon < 1$, we choose $\tau_1 = \tilde{\tau}$ and let $\tau_2 \rightarrow T$. On the other hand, if $\varepsilon > 1$, we choose $\tau_2 = \tilde{\tau}$ and let $\tau_1 \rightarrow 0$. Thus we get the estimate

$$\begin{aligned} & \int_0^T \int_{\Omega} m u^{m-2-\varepsilon} |\nabla u|^2 \zeta^2 dx dt + \frac{1}{\varepsilon|1-\varepsilon|} \operatorname{ess\,sup}_{t \in (0,T)} \int_{\Omega} u^{1-\varepsilon} \zeta^2 dx \\ & \leq \frac{C_1}{\varepsilon^2} \int_0^T \int_{\Omega} m u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \frac{C_2}{\varepsilon|1-\varepsilon|} \int_0^T \int_{\Omega} u^{1-\varepsilon} \zeta |\zeta_t| dx dt. \end{aligned}$$

□

Remark 2.5. Inequalities (2.3) and (2.7) hold even if the cut-off function at hand is not compactly supported in the time variable. This will be useful later on. For instance, it allows us to use cut-off functions, which vanish only on the parabolic boundary of the cylinder.

We now prove a measure theoretical lemma. The following lemma is a modification of the one in [4], as we work with balls instead of cubes. A similar statement has been presented in the metric space setting in [12]. However, for our purposes it is essential to keep track of the dependence of ε on the various constants. Hence we give a detailed proof for this known result.

Lemma 2.6. *Let $u \in W^{1,1}(B(x_0, \rho))$. Suppose that*

$$\begin{aligned} \|u\|_{W^{1,1}(B(x_0, \rho))} &\leq h \rho^{n-1} \quad \text{and} \\ |\{x \in B(x_0, \rho) : u(x) > 1\}| &\geq \tilde{\delta} |B(x_0, \rho)| \end{aligned}$$

for some $h > 0$ and $\tilde{\delta} \in (0, 1)$. Then for every $\delta, \lambda \in (0, 1)$ there exists $\tilde{x} \in B(x_0, \rho)$ and ε , such that

$$|\{x \in B(\tilde{x}, \varepsilon \rho) : u(x) > \lambda\}| > (1 - \delta) |B(\tilde{x}, \varepsilon \rho)|.$$

Here $\varepsilon = \frac{C(\lambda, \delta, n)}{h} \tilde{\delta}^2$.

Proof. Take $\varepsilon > 0$. Define

$$\mathcal{F} = \{B(x, \varepsilon \rho) : x \in B(x_0, (1 - \varepsilon)\rho)\}.$$

By Besicovitch's covering lemma, there is a constant c_n depending on the dimension n , and collections $\mathcal{G}_i \subset \mathcal{F}$, $i = 1, \dots, c_n$, such that each \mathcal{G}_i consists of disjoint balls and

$$B(x_0, (1 - \varepsilon)\rho) \subset \bigcup_{i=1}^{c_n} \bigcup_{B \in \mathcal{G}_i} B.$$

Moreover, the number of balls in the collection $\mathcal{G} = \bigcup_{i=1}^{c_n} \mathcal{G}_i$, denoted by $|\mathcal{G}|$, is at most $\frac{c_n}{\varepsilon^n}$. We observe, that

$$\begin{aligned} |\{x \in B(x_0, (1-\varepsilon)\rho) : u(x) > 1\}| &\geq |\{x \in B(x_0, (1-\varepsilon)\rho) : u(x) > 1\}| \\ &\quad - |B(x_0, \rho) \setminus B(x_0, (1-\varepsilon)\rho)| \\ &\geq (\tilde{\delta} - (1 - (1-\varepsilon)^n))|B(x_0, (1-\varepsilon)\rho)| \\ &\geq \frac{\tilde{\delta}}{2}|B(x_0, (1-\varepsilon)\rho)|, \end{aligned} \tag{2.8}$$

whenever $\varepsilon \leq \frac{\tilde{\delta}}{2^n}$. Define the subcollections

$$\begin{aligned} \mathcal{G}^+ &= \left\{ B \in \mathcal{G} : |\{x \in B : u(x) > 1\}| > \frac{\tilde{\delta}}{8c_n}|B| \right\} \quad \text{and} \\ \mathcal{G}^- &= \left\{ B \in \mathcal{G} : |\{x \in B : u(x) > 1\}| \leq \frac{\tilde{\delta}}{8c_n}|B| \right\}. \end{aligned}$$

Since \mathcal{G} is a covering of $B(x_0, (1-\varepsilon)\rho)$, (2.8) implies

$$\begin{aligned} \sum_{B_i \in \mathcal{G}^+} |\{x \in B_i : u(x) > 1\}| + \sum_{B_j \in \mathcal{G}^-} |\{x \in B_j : u(x) > 1\}| \\ \geq |\{x \in B(x_0, (1-\varepsilon)\rho) : u(x) > 1\}| \geq \frac{\tilde{\delta}}{2}|B(x_0, (1-\varepsilon)\rho)| \geq \frac{\tilde{\delta}}{4\varepsilon^n}|B| \end{aligned}$$

for every $B \in \mathcal{G}$. Here we assumed $(1-\varepsilon)^n \geq \frac{1}{2}$. Thus,

$$\begin{aligned} \frac{\tilde{\delta}}{4\varepsilon^n} &\leq \sum_{B_i \in \mathcal{G}^+} \frac{|\{x \in B_i : u(x) > 1\}|}{|B_i|} + \sum_{B_j \in \mathcal{G}^-} \frac{|\{x \in B_j : u(x) > 1\}|}{|B_j|} \\ &\leq |\mathcal{G}^+| + \frac{\tilde{\delta}}{8c_n}(|\mathcal{G}| - |\mathcal{G}^+|) \leq \left(1 - \frac{\tilde{\delta}}{8c_n}\right)|\mathcal{G}^+| + \frac{\tilde{\delta}}{8\varepsilon^n}. \end{aligned}$$

We get a lower bound for the number of cubes in \mathcal{G}^+ ,

$$|\mathcal{G}^+| > \frac{\tilde{\delta}c_n}{\varepsilon^n(8c_n - \tilde{\delta})}. \tag{2.9}$$

We fix the numbers $\delta, \lambda \in (0, 1)$. Suppose, that for every $B \in \mathcal{G}^+$, we have

$$|\{x \in B : u(x) > \lambda\}| \leq (1-\delta)|B|. \tag{2.10}$$

Then

$$|\{x \in B : u(x) \leq \lambda\}| > \delta|B|. \tag{2.11}$$

The De Giorgi type lemma [2, Lemma II] implies

$$(1-\lambda)|\{x \in B : u(x) \leq \lambda\}||\{x \in B : u(x) > 1\}| \leq C(\varepsilon\rho)^{n+1} \int_B |\nabla u| dx$$

Thus, using (2.11) and the fact, that $B \in \mathcal{G}^+$, gives the estimate

$$\frac{(1-\lambda)\delta\tilde{\delta}}{8c_n}|B|^2 \leq C(\varepsilon\rho)^{n+1} \int_B |\nabla u| dx.$$

We sum over \mathcal{G}^+ and use (2.9) to get

$$\frac{c(\lambda, \delta, n)\tilde{\delta}^2 \rho^{n-1}}{8c_n - \tilde{\delta}} \frac{1}{\varepsilon} \leq Cc_n \int_{B(x_0, \rho)} |\nabla u| dx \leq Cc_n h \rho^{n-1}.$$

We may choose

$$\varepsilon = \frac{C(\lambda, \delta, n)}{h} \tilde{\delta}^2,$$

thus leading to contradiction. Therefore (2.10) does not hold. That is, there exists $B \in \mathcal{G}^+$, such that

$$|\{x \in B : u(x) > \lambda\}| > (1 - \delta)|B|.$$

□

3. EXPANSION OF POSITIVITY

In this chapter, we will show that the supersolutions satisfy the “expansion of positivity”. That is, if the supersolution is large in a large portion of a ball at some time, then the supersolution will be bounded away from zero in a larger space-time cylinder after some waiting time. This is the statement of Lemma 3.7.

First, we will show, that if the supersolution u is large in a large portion of a ball at some time level, then u remains bounded away from zero in a large portion of the ball up to some time level.

Lemma 3.1. *Let $k > 0$ and $\gamma \in (0, 1)$. Suppose that u is a weak supersolution in Ω_{T_0} , such that*

$$|\{x \in B(x_0, \rho) : u(x, s) > k\}| \geq \gamma|B(x_0, \rho)|$$

for some $s \in (0, \rho^2)$. There exists a constant $C = C(m, n)$, such that

$$\left| \left\{ x \in B(x_0, \rho) : u(x, t) > \frac{\gamma}{8}k \right\} \right| \geq \frac{\gamma}{8}|B(x_0, \rho)|$$

for almost every $t \in \left(s, s + \frac{\gamma^2 \rho^2}{Ck^{m-1}}\right]$.

Proof. Denote $T = \frac{\gamma^2 \rho^2}{Ck^{m-1}}$ and $\varepsilon = \frac{\gamma}{5n}$. Let $\zeta \in C_0^\infty(B(x_0, (1+\varepsilon)\rho))$ be a cut-off function, such that

$$\begin{cases} \zeta = 1 & \text{in } B(x_0, \rho), \\ 0 \leq \zeta \leq 1 & \text{and} \\ |\nabla \zeta| \leq \frac{C_1}{\varepsilon \rho}. \end{cases}$$

Take $\tau_0, \tau \in (s, s + T)$, such that $\tau_0 < \tau$. Since u is a non-negative weak supersolution, (2.3) gives

$$\begin{aligned} & \int_{\tau_0}^{\tau} \int_{B(x_0, (1+\varepsilon)\rho)} u^{m-1} |\nabla(u - k)_-|^2 \zeta^2 dx dt \\ & + \int_{B(x_0, (1+\varepsilon)\rho)} \zeta(x)^2 (u(x, \tau) - k)_-^2 dx \\ & \leq C \int_{\tau_0}^{\tau} \int_{B(x_0, (1+\varepsilon)\rho)} u^{m-1} (u - k)_-^2 |\nabla \zeta|^2 dx dt \\ & + \int_{B(x_0, (1+\varepsilon)\rho)} \zeta(x)^2 (u(x, \tau_0) - k)_-^2 dx. \end{aligned}$$

This implies

$$\begin{aligned} & \int_{B(x_0, (1+\varepsilon)\rho)} \zeta(x)^2 (u(x, \tau) - k)_-^2 dx \\ & \leq C \int_{\tau_0}^{\tau} \int_{B(x_0, (1+\varepsilon)\rho)} u^{m-1} (u - k)_-^2 |\nabla \zeta|^2 dx dt \\ & + \int_{B(x_0, (1+\varepsilon)\rho)} \zeta(x)^2 (u(x, \tau_0) - k)_-^2 dx. \end{aligned}$$

Thus letting $\tau_0 \rightarrow s$ gives

$$\begin{aligned} & \int_{B(x_0, \rho)} \zeta(x)^2 (u(x, \tau) - k)_-^2 dx \\ & \leq C \int_s^{s+T} \int_{B(x_0, (1+\varepsilon)\rho)} u^{m-1} (u - k)_-^2 |\nabla \zeta|^2 dx dt \\ & + \int_{B(x_0, (1+\varepsilon)\rho)} \zeta(x)^2 (u(x, s) - k)_-^2 dx. \end{aligned} \tag{3.1}$$

This holds for almost every $\tau \in (s, s + T)$. We write

$$\begin{aligned} & \int_{B(x_0, (1+\varepsilon)\rho)} \zeta(x)^2 (u(x, s) - k)_-^2 dx \\ & = \int_{B(x_0, (1+\varepsilon)\rho) \setminus B(x_0, \rho)} \zeta(x)^2 (u(x, s) - k)_-^2 dx \\ & + \int_{B(x_0, \rho)} \zeta(x)^2 (u(x, s) - k)_-^2 dx. \end{aligned} \tag{3.2}$$

The first term on the right hand is bounded from above by

$$\begin{aligned} & \int_{B(x_0, (1+\varepsilon)\rho) \setminus B(x_0, \rho)} \zeta(x)^2 (u(x, s) - k)_-^2 dx \\ & \leq k^2 |B(x_0, (1+\varepsilon)\rho) \setminus B(x_0, \rho)| = k^2 ((1+\varepsilon)^n - 1) |B(x_0, \rho)|. \end{aligned}$$

By the assumption on u , the second term on the right hand side of (3.2) is bounded from above by

$$\begin{aligned} \int_{B(x_0, \rho)} \zeta(x)^2 (u(x, s) - k)_-^2 dx & \leq k^2 |\{x \in B(x_0, \rho) : u(x, s) \leq k\}| \\ & \leq (1 - \gamma) k^2 |B(x_0, \rho)|. \end{aligned}$$

Thus,

$$\begin{aligned} & \int_{B(x_0, (1+\varepsilon)\rho)} \zeta(x)^2 (u(x, s) - k)_-^2 dx \\ & \leq (((1+\varepsilon)^n - 1) + (1 - \gamma)) k^2 |B(x_0, \rho)| \leq \left(1 - \frac{3\gamma}{4}\right) k^2 |B(x_0, \rho)|. \end{aligned}$$

Here we used the fact

$$(1 + \varepsilon)^n - 1 \leq \frac{n\varepsilon}{1 - n\varepsilon} \leq \frac{\gamma}{4}.$$

The first term on the right hand side of (3.1) can be estimated using the boundedness of $|\nabla \zeta|$ and the fact that $\text{supp } \nabla \zeta \subset B(x_0, (1+\varepsilon)\rho) \setminus B(x_0, \rho)$. We have

$$\begin{aligned} & \int_s^{s+T} \int_{B(x_0, (1+\varepsilon)\rho)} u^{m-1} (u - k)_-^2 |\nabla \zeta|^2 dx dt \\ & \leq C_2 k^{m+1} \left(\frac{C_1}{\varepsilon \rho}\right)^2 T |B(x_0, (1+\varepsilon)\rho) \setminus B(x_0, \rho)| \\ & \leq C_2 k^{m+1} \left(\frac{C_1}{\varepsilon \rho}\right)^2 T \frac{\gamma}{4} |B(x_0, \rho)| = \frac{C_3 n^2 k^{m+1}}{\rho^2 \gamma} T |B(x_0, \rho)| \\ & = \frac{C_3 n^2 k^2 \gamma}{C} |B(x_0, \rho)| \leq \frac{\gamma k^2}{4} |B(x_0, \rho)|. \end{aligned}$$

Here C is chosen in such a way, that $C \geq 4C_3 n^2$. To get a lower bound for the left hand side of (3.1), we observe that

$$\left\{x \in B(x_0, \rho) : u(x, t) \leq \frac{\gamma}{8} k\right\} = \left\{x \in B(x_0, \rho) : (u(x, t) - k)_- \geq \left(1 - \frac{\gamma}{8}\right) k\right\}$$

and use Chebyshev's inequality to obtain

$$\begin{aligned} & \left(1 - \frac{\gamma}{8}\right)^2 k^2 \left|\left\{x \in B(x_0, \rho) : u(x, t) \leq \frac{\gamma}{8} k\right\}\right| \\ & \leq \int_{B(x_0, \rho)} \zeta(x)^2 (u(x, \tau) - k)_-^2 dx. \end{aligned}$$

Thus, collecting the facts gives

$$\left(1 - \frac{\gamma}{8}\right)^2 \left| \left\{ x \in B(x_0, \rho) : u(x, t) \leq \frac{\gamma}{8}k \right\} \right| \leq \left(1 - \frac{\gamma}{2}\right) |B(x_0, \rho)|.$$

Approximating

$$\frac{\left(1 - \frac{\gamma}{2}\right)}{\left(1 - \frac{\gamma}{8}\right)^2} \leq \frac{1 - \frac{\gamma}{2}}{1 - \frac{\gamma}{4}} \leq 1 - \frac{\gamma}{8}$$

then gives

$$|\{x \in B(x_0, \rho) : u(x, t) > \frac{\gamma}{8}k\}| \geq \frac{\gamma}{8}|B(x_0, \rho)|,$$

thus concluding the proof. \square

We will make a change of variables, in order to extend the positivity set even further in time. We consider the function $w(x, \tau) = g(\Lambda^{-1}(\tau))u(x, \Lambda^{-1}(\tau))$, which is a supersolution in a space-time cylinder $\Omega_{T'_0}$. The key idea here is to compensate the decay of u by a factor $g(\Lambda^{-1}(\tau))$, thus allowing us to consider the times beyond the threshold given by Lemma 3.1.

Lemma 3.2 (Change of Variables). *Let u be a non-negative weak supersolution in Ω_{T_0} , such that*

$$|\{x \in B(x_0, \rho) : u(x, s) > k\}| \geq \gamma|B(x_0, \rho)|.$$

Then

$$w(x, \tau) = \frac{e^{\frac{\tau}{m-1}}}{k} (\delta \rho^2)^{\frac{1}{m-1}} u \left(x, s + \frac{e^\tau}{k^{m-1}} \delta \rho^2 \right)$$

is a non-negative supersolution in $\Omega_{T'_0}$, where $T'_0 = \ln \left((T_0 - s) \frac{k^{m-1}}{\delta \rho^2} \right)$. Moreover, w satisfies

$$|\{x \in B(x_0, \rho) : w(x, \tau) > k_0\}| \geq \frac{\gamma}{8}|B(x_0, \rho)|$$

for almost every $\tau \geq 0$. Here $\delta = \frac{\gamma^2}{C}$ and $k_0 = \frac{\gamma}{8}(\delta \rho^2)^{\frac{1}{m-1}}$.

Proof. First we will show that w is indeed a weak supersolution. Define

$$g(t) = (t - s)^{\frac{1}{m-1}} \quad \text{and} \quad \Lambda(t) = \ln \left((t - s) \frac{k^{m-1}}{\delta \rho^2} \right).$$

Then w can be written as $w(x, \tau) = g(\Lambda^{-1}(\tau))u(x, \Lambda^{-1}(\tau))$. Let $\varphi \in C_0^\infty(\Omega_{T'_0})$ be a non-negative test function and define

$$\begin{aligned} \eta(x, t) &= g(t)\varphi(x, \Lambda(t)) = g(t)\tilde{\varphi}(x, t) \quad \text{and} \\ \tilde{w}(x, t) &= w(x, \Lambda(t)) = g(t)u(x, t). \end{aligned}$$

Now $\nabla u^m \cdot \nabla \eta$ and $u\eta_t$ can be written in terms of \tilde{w} , $\tilde{\varphi}$ and $\Lambda'(t)$ as

$$\nabla u^m \cdot \nabla \eta = \frac{1}{g^m} \nabla \tilde{w}^m \cdot \nabla \eta = \frac{1}{t - s} \nabla \tilde{w}^m \cdot \nabla \tilde{\varphi} = \Lambda'(t) \nabla \tilde{w}^m \cdot \nabla \tilde{\varphi} \quad \text{and}$$

$$u\eta_t = g'(t)u\tilde{\varphi} + \tilde{w}\tilde{\varphi}_t = \frac{g'}{g} \tilde{w}\tilde{\varphi} + \tilde{w}\tilde{\varphi}_t = \frac{\Lambda'(t)}{m-1} \tilde{w}\tilde{\varphi} + \tilde{w}\tilde{\varphi}_t.$$

Denote $T_1 = s + \frac{\delta \rho^2}{k^{m-1}}$. Since u is a weak supersolution, it satisfies

$$\begin{aligned} 0 &\leq \int_{T_1}^{T_0} \int_{\Omega} \left(-u\eta_t + \nabla u^m \cdot \nabla \eta \right) dx dt \\ &= - \int_{T_1}^{T_0} \int_{\Omega} \left(\frac{\Lambda'(t)}{m-1} \tilde{w} \tilde{\varphi} + \tilde{w} \tilde{\varphi}_t \right) dx dt + \int_{T_1}^{T_0} \int_{\Omega} \Lambda'(t) \nabla \tilde{w}^m \cdot \nabla \tilde{\varphi} dx dt. \end{aligned}$$

Recalling the definition of \tilde{w} and $\tilde{\varphi}$ gives

$$\begin{aligned} 0 &\leq - \int_{T_1}^{T_0} \int_{\Omega} \left(\frac{\Lambda'(t)}{m-1} w(x, \Lambda(t)) \varphi(x, \Lambda(t)) + w(x, \Lambda(t)) \varphi(x, \Lambda(t))_t \right) dx dt \\ &\quad + \int_{T_1}^{T_0} \int_{\Omega} \Lambda'(t) \nabla w(x, \Lambda(t))^m \cdot \nabla \varphi(x, \Lambda(t)) dx dt \\ &= - \int_0^{T'_0} \int_{\Omega} \left(\frac{1}{m-1} w \varphi + w \varphi_\tau \right) dx d\tau + \int_0^{T'_0} \int_{\Omega} \nabla w^m \cdot \nabla \varphi dx d\tau \\ &\leq \int_0^{T'_0} \int_{\Omega} \left(-w \varphi_\tau + \nabla w^m \cdot \nabla \varphi \right) dx d\tau, \end{aligned}$$

thus showing that w is a weak supersolution. The next step is to show, that w satisfies the inequality

$$|\{x \in B(x_0, \rho) : w(x, \tau) > k_0\}| \geq \frac{\gamma}{8} |B(x_0, \rho)|.$$

Take $\sigma \leq 1$. By assumption

$$|\{x \in B(x_0, \rho) : u(x, s) > \sigma k\}| \geq |\{x \in B(x_0, \rho) : u(x, s) > k\}| \geq \gamma |B(x_0, \rho)|.$$

By Lemma 3.1 we have

$$\left| \left\{ x \in B(x_0, \rho) : u(x, t) > \frac{\gamma}{8} \sigma k \right\} \right| \geq \frac{\gamma}{8} |B(x_0, \rho)|$$

for almost every $t \in \left(s, s + \frac{\delta \rho^2}{(\sigma k)^{m-1}} \right]$, where $\delta = \frac{\gamma^2}{C}$. Thus, in particular we have

$$\left| \left\{ x \in B(x_0, \rho) : u \left(x, s + \frac{\delta \rho^2}{(\sigma k)^{m-1}} \right) > \frac{\gamma}{8} \sigma k \right\} \right| \geq \frac{\gamma}{8} |B(x_0, \rho)|.$$

Choosing $\sigma = \sigma(\tau) = e^{-\frac{\tau}{m-1}}$ gives

$$\begin{aligned} &\left| \left\{ x \in B(x_0, \rho) : \frac{e^{\frac{\tau}{m-1}}}{k} (\delta \rho^2)^{\frac{1}{m-1}} u \left(x, s + \frac{\delta \rho^2}{k^{m-1}} e^\tau \right) > \frac{\gamma}{8} (\delta \rho^2)^{\frac{1}{m-1}} \right\} \right| \\ &\geq \frac{\gamma}{8} |B(x_0, \rho)| \end{aligned}$$

for almost every $\tau \geq 0$. Now, denoting $k_0 = \frac{\gamma}{8} (\delta \rho^2)^{\frac{1}{m-1}}$ and recalling the definition of w , shows that

$$|\{x \in B(x_0, \rho) : w(x, \tau) > k_0\}| \geq \frac{\gamma}{8} |B(x_0, \rho)|,$$

concluding the proof. \square

The next lemma shows, that w is small in a small portion of a space-time cylinder $B(x_0, 4\rho) \times (T, \theta T)$. This, however, realizes after a waiting time T , depending on γ and the size of the portion, where w is small. Then, the expansion of positivity for w follows from a De Giorgi type lemma for the weak supersolutions.

Lemma 3.3. *Let u be a weak supersolution in Ω_{T_0} and let w be defined as in Lemma 3.2. Then for every $\nu > 0$ there exists $\varepsilon > 0$ and a time level T , such that*

$$|\{(x, t) \in B(x_0, 4\rho) \times (T, \theta T) : w < \varepsilon k_0\}| \leq \nu |B(x_0, 4\rho) \times (T, \theta T)|.$$

The dependence of ε and T on the parameters ν and γ can be traced as

$$\varepsilon = 2^{-N} \quad \text{and} \quad T = \frac{2}{(\varepsilon k_0)^{m-1}} (4\rho)^2,$$

where $N = \left(\frac{C}{\gamma\nu}\right)^2 + 1$. The parameter $\theta \geq 2$ can be chosen as we please.

Proof. Let $k_j = 2^{-j} k_0$ for $j = 0, 1, \dots, N$ and $\varepsilon = 2^{-N}$, where $N \in \mathbb{N}$ will be determined in terms of γ and ν . By the De Giorgi type lemma [2, Lemma II] the following holds

$$\begin{aligned} & (k_j - k_{j+1}) |\{x \in B(x_0, 4\rho) : w(x, t) < k_{j+1}\}| \\ & \leq \tilde{C} \frac{\rho^{n+1}}{|\{x \in B(x_0, 4\rho) : w(x, t) > k_j\}|} \int_{A_j(t)} |\nabla w| dx \end{aligned}$$

at each time level t . Here $A_j(t) = \{x \in B(x_0, 4\rho) : k_{j+1} < w(x, t) < k_j\}$. By Lemma 3.2, we have

$$\begin{aligned} & |\{x \in B(x_0, 4\rho) : w(x, t) > k_j\}| \geq |\{x \in B(x_0, \rho) : w(x, t) > k_0\}| \\ & \geq \frac{\gamma}{8} |B(x_0, \rho)| = \gamma \tilde{C} \rho^n. \end{aligned}$$

Now, since $k_j - k_{j+1} = k_{j+1}$, we get the estimate

$$|\{x \in B(x_0, 4\rho) : w(x, t) < k_{j+1}\}| \leq \frac{C\rho}{\gamma k_{j+1}} \int_{A_j(t)} |\nabla w| dx.$$

Integrating over the time interval $(T, \theta T)$ gives

$$\begin{aligned} & |\{(x, t) \in B(x_0, 4\rho) \times (T, \theta T) : w(x, t) < k_{j+1}\}| \\ & \leq \frac{C\rho}{\gamma k_{j+1}} \int_T^{\theta T} \int_{A_j(t)} |\nabla w| dx dt. \end{aligned} \tag{3.3}$$

In order to control the right hand side, we denote $A_j = \{(x, t) \in B(x_0, 4\rho) \times (T, \theta T) : k_{j+1} < w < k_j\}$ and use Hölder's inequality to get

$$\iint_{A_j} |\nabla w| dx dt \leq \left(\iint_{A_j} |\nabla w|^2 dx dt \right)^{1/2} |A_j|^{1/2}. \tag{3.4}$$

Since $k_{j+1} < w < k_j$ in A_j , we may approximate

$$\begin{aligned}
\iint_{A_j} |\nabla w|^2 dx dt &= \iint_{A_j} |\nabla(w - k_j)_-|^2 dx dt \\
&\leq \frac{1}{k_{j+1}^{m-1}} \iint_{A_j} w^{m-1} |\nabla(w - k_j)_-|^2 dx dt \\
&\leq \frac{1}{k_{j+1}^{m-1}} \int_T^{\theta T} \int_{B(x_0, 4\rho)} w^{m-1} |\nabla(w - k_j)_-|^2 dx dt.
\end{aligned} \tag{3.5}$$

Let $\zeta \in C_0^\infty(B(x_0, 8\rho) \times (0, T'_0))$ be a smooth cut-off function, such that $0 \leq \zeta \leq 1$ and

$$\begin{cases} \zeta = 1 & \text{in } B(x_0, 4\rho) \times (T, \theta T), \\ |\nabla \zeta| \leq \frac{\tilde{C}}{4\rho} & \text{and} \\ |\zeta_t| \leq \frac{\tilde{C}}{T}. \end{cases}$$

Using Lemma 2.2, we get the estimate

$$\begin{aligned}
&\frac{1}{k_{j+1}^{m-1}} \int_T^{\theta T} \int_{B(x_0, 4\rho)} w^{m-1} |\nabla(w - k_j)_-|^2 dx dt \\
&\leq \frac{C}{k_{j+1}^{m-1}} \int_0^{\theta T} \int_{B(x_0, 8\rho)} \left((w - k_j)_-^2 |\zeta_t| + w^{m-1} (w - k_j)_-^2 |\nabla \zeta|^2 \right) dx dt \\
&\leq \frac{C}{k_{j+1}^{m-1}} \left(\frac{k_j^2}{T} + \frac{k_j^{m+1}}{(4\rho)^2} \right) |B(x_0, 8\rho) \times (0, \theta T)| \\
&\leq \frac{C k_j^{m+1}}{k_{j+1}^{m-1} (4\rho)^2} \left(\frac{2}{k_j^{m-1} (\varepsilon k_0)^{m-1}} + 1 \right) |B(x_0, 4\rho) \times (T, \theta T)| \\
&\leq \frac{C k_j^2}{(4\rho)^2} |B(x_0, 4\rho) \times (T, \theta T)|
\end{aligned} \tag{3.6}$$

Combining the estimates from (3.3), (3.4), (3.5) and (3.6) gives

$$\begin{aligned}
&|\{(x, t) \in B(x_0, 4\rho) \times (T, \theta T) : w(x, t) < k_{j+1}\}| \\
&\leq \frac{C}{\gamma} |A_j|^{1/2} |B(x_0, 4\rho) \times (T, \theta T)|^{1/2}.
\end{aligned}$$

Since, $k_N < k_{j+1}$ for $j = 0, \dots, N-1$, we have

$$\begin{aligned}
&|\{(x, t) \in B(x_0, 4\rho) \times (T, \theta T) : w(x, t) < k_N\}|^2 \\
&\leq \frac{C}{\gamma^2} |A_j| |B(x_0, 4\rho) \times (T, \theta T)|.
\end{aligned}$$

By definition, the sets $A_j \subset B(x_0, 4\rho) \times (T, \theta T)$ are disjoint, and therefore summing over j gives

$$\begin{aligned} & (N-1)|\{(x, t) \in B(x_0, 4\rho) \times (T, \theta T) : w(x, t) < k_N\}|^2 \\ & \leq \frac{C}{\gamma^2}|B(x_0, 4\rho) \times (T, \theta T)|^2. \end{aligned}$$

Hence, the result holds for $\nu = \frac{C}{\gamma\sqrt{N-1}}$ and $\varepsilon = k_N$, where $N \in \mathbb{N}$ can be chosen as we please. \square

We will prove a De Giorgi type lemma for the weak supersolutions.

Lemma 3.4. *Let u be a non-negative, locally bounded weak supersolution in a neighbourhood of $U_{2\rho} = B(x_0, 2\rho) \times (t_0, t_0 + \lambda(2\rho)^2)$. Let $\xi, a \in (0, 1)$ and let $\mu \geq \text{ess sup}_{U_{2\rho}} u$. Then, there exists a constant $\nu = \nu(a, \xi, \mu, \lambda, m, n)$, such that if*

$$|\{(x, t) \in U_{2\rho} : u(x, t) \leq \xi\mu\}| \leq \nu|U_{2\rho}|,$$

then

$$u \geq a\xi\mu \quad \text{almost everywhere in } B(x_0, \rho) \times (t_0 + 3\lambda\rho^2, t_0 + 4\lambda\rho^2).$$

Proof. Denote $T = t_0 + 4\lambda\rho^2$. Let $\rho_j = (1 + 2^{-j})\rho$, $T_j = T - \lambda\rho_j^2$, $B^j = B(x_0, \rho_j)$ and $U^j = B^j \times (T_j, T)$. Moreover, let $k_j = (2^{-j} + (1 - 2^{-j})a)\xi\mu$. Define a function $v = \max\{u, \frac{1}{2}a\xi\mu\}$. We observe, that $k_j > \frac{1}{2}a\xi\mu$, which implies

$$A_j = \{(x, t) \in U^j : v(x, t) < k_j\} = \{(x, t) \in U^j : u(x, t) < k_j\}.$$

Thus, it suffices to show, that $|A_j| \rightarrow 0$ as $j \rightarrow \infty$. We will show this by using the fast geometric convergence lemma [8, Lemma 7.1, p.220].

On the set A_{j+1} we have $v < k_{j+1}$ and therefore

$$(v - k_j)_- > k_j - k_{j+1} = \frac{1-a}{2^{j+1}}\xi\mu.$$

Let ζ be a smooth cut-off function, such that $0 \leq \zeta \leq 1$ and

$$\begin{cases} \zeta = 1 & \text{in } U^{j+1}, \\ \zeta = 0 & \text{on } \partial_p U^j, \\ |\nabla \zeta| \leq \frac{1}{\rho_j - \rho_{j+1}} = \frac{2^{k+1}}{\rho} & \text{and} \\ |\zeta_t| \leq \frac{1}{T_j - T_{j+1}} \leq \frac{2^{2(j+1)}}{\lambda\rho^2}. \end{cases}$$

Now

$$\left(\frac{1-a}{2^{j+1}}\right)^2 (\xi\mu)^2 |A_{j+1}| \leq \iint_{U^{j+1}} (v - k_j)_-^2 dx dt \leq \iint_{U^j} (v - k_j)_-^2 \zeta^2 dx dt. \quad (3.7)$$

Using Hölder's inequality and the parabolic Sobolev's inequality [3, Proposition 3.1] with $q = 2\frac{n+2}{n}$, $p = 2$ and $m = 2$, we get

$$\begin{aligned}
& \iint_{U^j} (v - k_j)_-^2 \zeta^2 dx dt \leq \left(\iint_{U^j} ((v - k_j)_-^2 \zeta^2)^{(n+2)/n} dx dt \right)^{n/(n+2)} |A_j|^{2/(n+2)} \\
& \leq C \left(\iint_{U^j} |\nabla((v - k_j)_- \zeta)|^2 dx dt \right)^{n/(n+2)} \\
& \times \left(\operatorname{ess\,sup}_{t \in (T_j, T)} \int_{B^j} (v - k_j)_-^2 \zeta^2 dx \right)^{2/(n+2)} |A_j|^{2/(n+2)}.
\end{aligned} \tag{3.8}$$

To find an upper bound for the right hand side, we observe

$$\begin{aligned}
& \left(\frac{a\xi\mu}{2} \right)^{m-1} \iint_{U^j} |\nabla((v - k_j)_- \zeta)|^2 dx dt \leq \iint_{U^j} v^{m-1} |\nabla((v - k_j)_- \zeta)|^2 dx dt \\
& = \iint_{\{(x,t) \in U^j : u(x,t) = v(x,t)\}} u^{m-1} |\nabla((u - k_j)_- \zeta)|^2 dx dt \\
& + \iint_{\{(x,t) \in U^j : u(x,t) < v(x,t)\}} v^{m-1} |\nabla((v - k_j)_- \zeta)|^2 dx dt = I_1 + I_2.
\end{aligned}$$

Now I_1 can be estimated using Lemma 2.2 as

$$\begin{aligned}
I_1 & \leq C \iint_{U^j} \left(u^{m-1} |\nabla(u - k_j)_-|^2 \zeta^2 + u^{m-1} (u - k_j)_-^2 |\nabla \zeta|^2 \right) dx dt \\
& \leq C \iint_{U^j} \left((u - k_j)_-^2 \zeta |\zeta_t| + u^{m-1} (u - k_j)_-^2 |\nabla \zeta|^2 \right) dx dt \\
& \leq C \left(\frac{k_j^2 2^{2(j+1)}}{\lambda \rho^2} + \frac{2^{2(j+1)}}{k_j^{m+1} \rho^2} \right) |A_j| \\
& \leq \frac{C(\xi\mu)^{m+1} 2^{2j}}{\rho^2} \left(1 + \frac{1}{\lambda(\xi\mu)^{m-1}} \right) |A_j|.
\end{aligned}$$

On the set $\{(x, t) \in U^j : u(x, t) < v(x, t)\}$, we have $v = \frac{a\xi\mu}{2} \leq \xi\mu$. Therefore I_2 can be approximated by

$$\begin{aligned}
I_2 & \leq (\xi\mu)^{m-1} \iint_{\{(x,t) \in U^j : u(x,t) < v(x,t)\}} (v - k_j)_-^2 |\nabla \zeta|^2 dx dt \\
& \leq \frac{(\xi\mu)^{m-1} k_j^2 2^{2(j+1)}}{\rho^2} |A_j| \\
& \leq \frac{C(\xi\mu)^{m+1} 2^{2j}}{\rho^2} |A_j|.
\end{aligned}$$

Since $v \geq u$, we have $(u - k_j)_- \geq (v - k_j)_-$ and thus we may use Lemma 2.2 and the same reasoning as for the upper bound of I_1 to get,

$$\operatorname{ess\,sup}_{t \in (T_j, T)} \int_{B^j} (v - k_j)_-^2 \zeta^2 dx \leq \frac{C(\xi\mu)^{m+1} 2^{2j}}{\rho^2} \left(1 + \frac{1}{\lambda(\xi\mu)^{m-1}}\right) |A_j|.$$

Collecting the facts, (3.7) and (3.8) show that

$$\begin{aligned} & \left(\frac{1-a}{2^{j+1}}\right)^2 (\xi\mu)^2 |A_{j+1}| \\ & \leq \left(\frac{C 2^{2j} (\xi\mu)^{m+1}}{(\frac{1}{2}a)^{m-1} \rho^2 (\xi\mu)^{m-1}} \left(1 + \frac{1}{\lambda(\xi\mu)^{m-1}}\right) |A_j|\right)^{n/(n+2)} \\ & \times \left(\frac{C 2^{2j} (\xi\mu)^{m+1}}{(\frac{1}{2}a)^{m-1} \rho^2} \left(1 + \frac{1}{\lambda(\xi\mu)^{m-1}}\right) |A_j|\right)^{2/(n+2)} |A_j|^{2/(n+2)} \\ & = \frac{C}{(\frac{1}{2}a)^{m-1}} \frac{2^{2j}}{\rho^2} (\xi\mu)^{(2n+2(m+1))/(n+2)} \left(1 + \frac{1}{\lambda(\xi\mu)^{m-1}}\right) |A_j|^{1+2/(n+2)}. \end{aligned}$$

This can be written as

$$|A_{j+1}| \leq \frac{C}{(\frac{1}{2}a)^{m-1} (1-a)^2} \frac{2^{4j}}{\rho^2} \left(\frac{\lambda(\xi\mu)^{m-1} + 1}{\lambda(\xi\mu)^{((m-1)n)/(n+2)}}\right) |A_j|^{1+2/(n+2)}.$$

We denote $Y_j = \frac{|A_j|}{|U^j|}$. Next, We divide both sides by $\rho^{n+2}\lambda$ and observe, that $\rho^2 \lambda^{2/(n+2)} \geq C |U^j|^{2/(n+2)}$ to get

$$Y_{j+1} \leq 2^{4j} \frac{C}{(\frac{1}{2}a)^{m-1} (1-a)^2} \left(\frac{\lambda(\xi\mu)^{m-1} + 1}{(\lambda(\xi\mu)^{m-1})^{n/(n+2)}}\right) Y_j^{1+2/(n+2)}.$$

Now by fast geometric convergence [8, Lemma 7.1, p.220], $Y_j \rightarrow 0$, if

$$Y_0 \leq \left(\frac{(\frac{1}{2}a)^{m-1} (1-a)^2}{C(\lambda(\xi\mu)^{m-1} + 1)}\right)^{(n+2)/2} (\lambda(\xi\mu)^{m-1})^{n/2} 2^{-(n+2)^2}.$$

Choosing ν to be the quantity on the right hand side, this holds by the assumption on u . Thus $Y_j \rightarrow 0$ as $j \rightarrow \infty$, implying that $u \geq a\xi\mu$ almost everywhere in $B(x_0, \rho) \times (t_0 + 3\lambda\rho^2, t_0 + 4\lambda\rho^2)$. \square

Now, we have all the necessary tools for showing, that the expansion of positivity holds for w . We observe, that in the following lemma, we can make ε as small as we please by choosing θ larger. This, however, increases the waiting time. This turns out to be a useful property in the proof of the main theorem.

Lemma 3.5 (Expansion of positivity for w). *Let u be a non-negative weak supersolution in Ω_{T_0} , such that*

$$|\{x \in B(x_0, \rho) : u(x, s) > k\}| \geq \gamma |B(x_0, \rho)|$$

at some time level $s \in (0, \rho^2)$ for some $\gamma \in (0, 1)$ and let w be defined as in Lemma 3.2. Then there exists $\varepsilon > 0$, depending only on m, n, γ and θ , such that

$$w \geq \frac{1}{2}\varepsilon k_0 \text{ almost everywhere in } B(x_0, 2\rho)$$

for almost every

$$t \in \left(\frac{1+3\theta}{4} \frac{C}{(\varepsilon k_0)^{m-1}} (2\rho)^2, \theta \frac{C}{(\varepsilon k_0)^{m-1}} (2\rho)^2 \right).$$

Proof. Let $a = \frac{1}{2}$ and suppose, that ξ is chosen in such a way, that $\xi\mu = \varepsilon k_0$. The claim holds by Lemma 3.4, if

$$|\{(x, t) \in U_{4\rho} : w < \varepsilon k_0\}| \leq \nu |U_{4\rho}|, \quad (3.9)$$

where the constants t_0 and λ are chosen in such a way, that

$$U_{4\rho} = B(x_0, 4\rho) \times \left(\frac{2}{(\varepsilon k_0)^{m-1}} (4\rho)^2, \frac{2\theta}{(\varepsilon k_0)^{m-1}} (4\rho)^2 \right)$$

and

$$\nu = \left(\frac{1}{C 4^m (2\theta - 1)} \right)^{(n+2)/2} (2(\theta - 1))^{n/2} 2^{-(n+2)^2}.$$

By Lemma 3.3, we can choose ε in such a way, that (3.9) holds. We observe, that ε depends only on m, n, γ and θ . □

Now, we will return to the original coordinates and show, that the expansion of positivity holds for u as well.

Lemma 3.6 (Expansion of positivity for u). *Let u be a weak supersolution in Ω_{T_0} , such that*

$$|\{x \in B(x_0, \rho) : u(x, s) > k\}| \geq \gamma |B(x_0, \rho)|.$$

at some time level $s \in (0, \rho^2)$ for some $\gamma \in (0, 1)$. Then

$$u \geq \eta k \text{ almost everywhere in } B(x_0, 2\rho) \times \left(s + \frac{1}{2} \frac{b^{m-1}}{(k\eta)^{m-1}} \delta \rho^2, s + \frac{b^{m-1}}{(k\eta)^{m-1}} \delta \rho^2 \right),$$

here

$$b = \frac{\varepsilon \gamma}{16},$$

$$\eta = \frac{b}{b_1^\theta} \text{ and}$$

$$b_1 = \exp \left(\frac{C}{(m-1)(\varepsilon \gamma)^{m-1} \delta} \right).$$

Proof. By Lemma 3.5,

$$w(\cdot, \tau) \geq \frac{1}{2}\varepsilon k_0 \text{ for almost every } \tau \in \left(\frac{1+3\theta}{4} \frac{C}{(\varepsilon k_0)^{m-1}} (2\rho)^2, \theta \frac{C}{(\varepsilon k_0)^{m-1}} (2\rho)^2 \right).$$

Recalling the definition of $k_0 = \frac{\gamma}{8}(\delta\rho^2)^{\frac{1}{m-1}}$, this states

$$\tau \in \left(\frac{1+3\theta}{4} \frac{C}{(\varepsilon\gamma)^{m-1}\delta}, \theta \frac{C}{(\varepsilon\gamma)^{m-1}\delta} \right)$$

and therefore

$$\begin{aligned} e^{\frac{\tau}{m-1}} &\in \left(\exp \left(\frac{1+3\theta}{4} \frac{C}{(m-1)(\varepsilon\gamma)^{m-1}\delta} \right), \exp \left(\theta \frac{C}{(m-1)(\varepsilon\gamma)^{m-1}\delta} \right) \right) \\ &= (b_1^{\frac{1+3\theta}{4}}, b_1^\theta). \end{aligned}$$

Recalling the definition of w , we get the estimate

$$w(x, \tau) = \frac{e^{\frac{\tau}{m-1}}}{k} (\delta\rho^2)^{\frac{1}{m-1}} u(x, \Lambda^{-1}(\tau)) \leq \frac{b_1^\theta}{k} (\delta\rho^2)^{\frac{1}{m-1}} u(x, \Lambda^{-1}(\tau)),$$

where $\Lambda^{-1}(\tau)$ is defined as in Lemma 3.2. Thus

$$u(x, t) \geq \eta k \text{ for almost every } t \in \left(s + \frac{b_1^{\frac{1+3\theta}{4}(m-1)}}{k^{m-1}} \delta\rho^2, s + \frac{b_1^{\theta(m-1)}}{k^{m-1}} \delta\rho^2 \right),$$

where $\eta = \frac{\varepsilon\gamma}{16b_1^\theta}$. Choosing $b = \frac{\varepsilon\gamma}{16}$ concludes the proof. \square

Finally, we prove a refined version of Lemma 3.6. The crucial feature in the following lemma is the power-like dependency of η on the parameter γ , whereas in Lemma 3.6 the dependency is exponential. The idea of the proof is as follows. We use the measure theoretical lemma (Lemma 2.6) to find a small ball $B(\tilde{x}, \varepsilon\rho)$, where u is large in a fixed portion of the ball. Then we may use Lemma 3.6 iteratively to get the result.

Lemma 3.7. *Let $u \geq 0$ be a weak supersolution, such that*

$$|\{x \in B(x_0, \rho) : u(x, s) > k\}| \geq \gamma |B(x_0, \rho)|$$

at some time level $s \in (0, \rho^2)$ for some $\gamma \in (0, 1)$. Then, there exist constants $\eta_0, \delta \in (0, 1)$, $b, d > 1$ and a time level $\tilde{t} \in \left(s + \frac{1}{2} \frac{\delta\rho^2}{k^{m-1}}, s + \frac{\delta\rho^2}{k^{m-1}} \right)$, such that

$$u(\cdot, t) \geq \eta k \quad \text{almost everywhere in } B(x_0, 2\rho)$$

for almost every $t \in \left(\tilde{t} + \frac{1}{2} \frac{b^{m-1}}{(\eta k)^{m-1}} \delta\rho^2, \tilde{t} + \frac{b^{m-1}}{(\eta k)^{m-1}} \delta\rho^2 \right)$. Here $\eta = \eta_0 \gamma^d$.

Proof. Denote $T = \frac{\delta}{k^{m-1}} \rho^2$. Let $U_\rho = B(x_0, \rho) \times (s + \frac{1}{2}T, s + T)$ and let $\tilde{U}_\rho = B(x_0, 2\rho) \times (s, s + T)$. Choose a smooth cut-off function ζ , such that $0 \leq \zeta \leq 1$ and

$$\begin{cases} \zeta = 1 & \text{in } U_\rho, \\ \zeta = 0 & \text{on } \partial_p \tilde{U}_\rho, \\ |\nabla \zeta| \leq \frac{C}{\rho} & \text{and} \\ |\zeta_t| \leq \frac{C}{T}. \end{cases}$$

We observe, that

$$\begin{aligned} \iint_{U_\rho \cap \{u < k\}} |\nabla u^{\frac{m+1}{2}}|^2 dx dt &= \iint_{U_\rho \cap \{u < k\}} u^{m-1} |\nabla u|^2 dx dt \\ &\leq \iint_{\tilde{U}_\rho} u^{m-1} |\nabla(u-k)_-|^2 \zeta^2 dx dt. \end{aligned} \tag{3.10}$$

By the Caccioppoli estimate in Lemma 2.2, we have

$$\begin{aligned} &\iint_{\tilde{U}_\rho} u^{m-1} |\nabla(u-k)_-|^2 \zeta^2 dx dt \\ &\leq C \left(\iint_{\tilde{U}_\rho} (u-k)_-^2 \zeta |\zeta_t| + u^{m-1} (u-k)_-^2 |\nabla \zeta|^2 dx dt \right) \\ &\leq \frac{C k^{m+1} |\tilde{U}_\rho|}{\delta \rho^2}. \end{aligned}$$

Thus, Hölder's inequality, together with (3.10), gives

$$\iint_{U_\rho \cap \{u < k\}} |\nabla u^{\frac{m+1}{2}}| dx dt \leq \frac{C k^{\frac{m+1}{2}} |U_\rho|}{\gamma \rho}$$

Define a function

$$w = \frac{(u^{\frac{m+1}{2}} - k^{\frac{m+1}{2}})_-}{k^{\frac{m+1}{2}}}.$$

Now

$$\iint_{U_\rho} |\nabla w| dx dt = \frac{1}{k^{\frac{m+1}{2}}} \iint_{U_\rho \cap \{u < k\}} |\nabla u^{\frac{m+1}{2}}| dx dt \leq \frac{C |U_\rho|}{\gamma \rho}.$$

Thus, we have

$$\frac{2}{T} \iint_{U_\rho} |\nabla w| dx dt \leq \frac{C}{\gamma} \rho^{n-1}$$

and therefore we can find $\tilde{t} \in (s + \frac{1}{2}T, s + T)$, such that

$$\int_{B(x_0, \rho)} |\nabla w(x, \tilde{t})| dx \leq \frac{C}{\gamma} \rho^{n-1}.$$

On the other hand, by Lemma 3.1, we have

$$\left| \left\{ x \in B(x_0, \rho) : u(x, t) > \frac{\gamma}{8} k \right\} \right| \geq \frac{\gamma}{8} |B(x_0, \rho)|$$

for almost every $t \in \left(s, s + \frac{\gamma^2 \rho^2}{C k^{m-1}} \right]$. We observe, that whenever $u > \frac{\gamma}{8} k$, we have

$$w = \frac{(u^{\frac{m+1}{2}} - k^{\frac{m+1}{2}})_-}{k^{\frac{m+1}{2}}} < 1 - \left(\frac{\gamma}{8} \right)^{\frac{m+1}{2}}.$$

Define a function

$$v = \frac{1-w}{\left(\frac{\gamma}{8} \right)^{\frac{m+1}{2}}}.$$

Now v has the following properties:

$$|\{x \in B(x_0, \rho) : v(x, t) > 1\}| \geq \frac{\gamma}{8} |B(x_0, \rho)|$$

for almost every $t \in (s + \frac{1}{2}T, s + T)$ and there exists $\tilde{t} \in (s + \frac{1}{2}T, s + T)$, such that

$$\int_{B(x_0, \rho)} |\nabla v| dx = \frac{1}{\left(\frac{\gamma}{8}\right)^{\frac{m+1}{2}}} \int_{B(x_0, \rho)} |\nabla w| dx \leq \frac{C}{\gamma \left(\frac{\gamma}{8}\right)^{\frac{m+1}{2}}} \rho^{n-1}.$$

By Lemma 2.6 with constants $\delta = \frac{1}{2}$ and $\lambda = \frac{1}{2^{\frac{m+1}{2}}}$, we find a ball $B(\tilde{x}, \varepsilon\rho)$, such that

$$\left| \left\{ x \in B(\tilde{x}, \varepsilon\rho) : v > \frac{1}{2^{\frac{m+1}{2}}} \right\} \right| > \frac{1}{2} |B(\tilde{x}, \varepsilon\rho)|.$$

Here $\varepsilon = C \left(\frac{\gamma}{8}\right)^{2+\frac{m+1}{2}} \gamma$. We observe, that whenever $v > \frac{1}{2^{\frac{m+1}{2}}}$, we have $w < 1 - \left(\frac{\gamma}{16}\right)^{\frac{m+1}{2}}$ and thus $u > \frac{\gamma}{16}k$. Therefore

$$\left| \left\{ x \in B(\tilde{x}, \varepsilon\rho) : u(x, \tilde{t}) > \frac{\gamma}{16}k \right\} \right| \geq \frac{1}{2} |B(\tilde{x}, \varepsilon\rho)| \quad (3.11)$$

at some time $\tilde{t} \in \left(s + \frac{1}{2} \frac{\delta \rho^2}{k^{m-1}}, s + \frac{\delta \rho^2}{k^{m-1}}\right)$. Denote

$$T_i = \frac{\bar{b}^{m-1}}{(\bar{\eta}^i \tilde{k})^{m-1}} \bar{\delta} (2^{i-1} \varepsilon \rho)^2,$$

where the constants \bar{b} , $\bar{\eta}$ and $\bar{\delta}$ correspond to $\gamma = \frac{1}{2}$ in Lemma 3.6 and $\tilde{k} = \frac{\gamma}{16}k$. Applying Lemma 3.6 to (3.11) shows, that

$$u(x, t) \geq \tilde{\eta} \tilde{k} \quad \text{almost everywhere in } B(\tilde{x}, 2\varepsilon\rho)$$

for almost every $t_1 \in (\tilde{t} + \frac{1}{2}T_1, \tilde{t} + T_1)$. Applying Lemma 3.6 iteratively shows that

$$u \geq \bar{\eta}^i \tilde{k} \quad \text{almost every where in } B(\tilde{x}, 2^i \varepsilon\rho)$$

for almost every $t_i \in (t_{i-1} + \frac{1}{2}T_i, t_{i-1} + T_i)$. Without loss of generality, we may assume $2^N \varepsilon = 4$ for some $N \in \mathbb{N}$. Thus, we obtain

$$u \geq \bar{\eta}^N \tilde{k} \quad \text{almost everywhere in } B(\tilde{x}, 4\rho)$$

for almost every

$$t \in \left(\tilde{t} + \frac{1}{2} \frac{\bar{b}^{m-1}}{\tilde{k}^{m-1}} \bar{\delta} (\varepsilon\rho)^2 \sum_{i=1}^N \frac{4^{i-1}}{\bar{\eta}^{i(m-1)}}, \tilde{t} + \frac{\bar{b}^{m-1}}{\tilde{k}^{m-1}} \bar{\delta} (\varepsilon\rho)^2 \sum_{i=1}^N \frac{4^{i-1}}{\bar{\eta}^{i(m-1)}} \right).$$

This implies

$$u \geq \bar{\eta}^N \tilde{k} \quad \text{almost everywhere in } B(x_0, 2\rho)$$

for almost every

$$t \in \left(\tilde{t} + \frac{2}{3} \frac{\bar{b}^{m-1}}{(\tilde{k} \bar{\eta}^N)^{m-1}} \bar{\delta} (2\rho)^2, \tilde{t} + \frac{5}{6} \frac{\bar{b}^{m-1}}{(\tilde{k} \bar{\eta}^N)^{m-1}} \bar{\delta} (2\rho)^2 \right).$$

Since $2^N \varepsilon = 4$, we may write $N = 2 + \log_{\bar{\eta}} \varepsilon^{-\frac{\ln \bar{\eta}}{\ln 2}}$. Recalling the definition of ε and \tilde{k} , we have

$$\bar{\eta}^N \tilde{k} = \eta_0 \gamma^d k,$$

where $d = -\frac{\ln \bar{\eta}}{\ln 2} \left(3 + \frac{m+1}{2}\right) + 1$ and η_0 is a constant depending only on m, n and θ . Therefore, choosing suitable constants b and δ gives

$$u \geq \eta_0 \gamma^d k \quad \text{almost everywhere in } B(x_0, 2\rho)$$

for almost every

$$t \in \left(\tilde{t} + \frac{1}{2} \frac{b^{m-1}}{(\eta_0 \gamma^d k)^{m-1}} \delta \rho^2, \tilde{t} + \frac{b^{m-1}}{(\eta_0 \gamma^d k)^{m-1}} \delta \rho^2 \right),$$

thus concluding the proof. \square

4. THE COLD ALTERNATIVE

We will show, that if the supersolution u is large only in a small portion of the ball $B(x_0, \rho)$ at every time level $t \in (0, \rho^2)$, then u is bounded away from zero after some waiting time, provided that the integral average of u over the ball is large enough at time $t = 0$. The strategy of the proof is the following. We will use a qualitative version of a reverse Hölder's inequality to show, that the L^q -norms of supersolutions over cylinders of radius $\frac{3}{4}\rho$ are uniformly bounded. Then, using the Caccioppoli estimates together with the boundedness of L^q -norms, we show that the L^1 -norms of ∇u^m are bounded as well, thus giving us a uniform lower bound for the integral averages of u over a smaller ball $B(x_0, \frac{5}{8}\rho)$, provided that the integral average at $t = 0$ is large enough. Finally, we use a real analytic lemma (Lemma 4.5) to find a time level $\tau \in \left(0, \left(\frac{5}{8}\rho\right)^2\right)$, such that

$$\left| \left\{ x \in B\left(x_0, \frac{5}{8}\rho\right) : u(x, \tau) > C_1 \right\} \right| \geq C_2 \left| B\left(x_0, \frac{5}{8}\rho\right) \right|,$$

and thus the boundedness from below follows from Lemma 3.7.

First, we will prove a qualitative version of a reverse Hölder's inequality for the weak supersolutions of the porous medium equation.

Lemma 4.1. *Let u be a weak supersolution in a neighbourhood of $B(x_0, \rho) \times (0, \rho^2)$, such that $u > 0$. Let $q \in (m-1, m + \frac{2}{n})$ and let s be defined as*

$$s = (m-1) + \left(1 + \frac{2}{n}\right)^{-(N+1)} (q - (m-1)),$$

where $N \in \mathbb{N}$. If

$$\int_0^{\rho^2} \int_{B(x_0, \rho)} u^s dx dt \leq \tilde{C},$$

for some \tilde{C} , then

$$\int_0^{(\alpha\rho)^2} \int_{B(x_0, \alpha\rho)} u^q dx dt \leq C$$

for every $\alpha \in (\frac{1}{2}, 1)$. Here $C = C(m, n, q, N, \tilde{C}, \alpha)$.

Proof. Fix $\alpha \in (\frac{1}{2}, 1)$ and $N \in \mathbb{N}$. Define

$$r_j = \rho - (1 - \alpha)\rho \frac{1 - 2^{-j}}{1 - 2^{-(N+1)}}.$$

Then $r_0 = \rho$ and $r_{N+1} = \alpha\rho$. Denote $B^j = B(x_0, r_j)$ and $U^j = B^j \times (0, r_j^2)$. For fixed j , let ζ be a smooth cut-off function, such that $0 \leq \zeta \leq 1$ and

$$\begin{aligned} \zeta &= 1 \quad \text{in } U^{j+1}, \\ \zeta &= 0 \quad \text{on } \partial^p U^j, \\ |\nabla \zeta| &\leq \frac{C}{r_j - r_{j+1}} \leq \frac{C2^{j+1}}{(1 - \alpha)\rho} \quad \text{and} \\ |\zeta_t| &\leq \frac{C}{r_j^2 - r_{j+1}^2} \leq \frac{C2^{2(j+1)}}{((1 - \alpha)\rho)^2}. \end{aligned}$$

In order to utilize the Caccioppoli estimates, we choose

$$\begin{aligned} a &= m - \varepsilon, \\ \kappa &= 1 + \frac{2(1 - \varepsilon)}{n(m - \varepsilon)} \quad \text{and} \\ b &= 2\frac{m - \varepsilon}{1 - \varepsilon}, \end{aligned}$$

where $\varepsilon \in (0, 1)$, and use the parabolic Sobolev's inequality [3, Proposition 3.1] with $q = 2\kappa$, $p = 2$ and $m = n(\kappa - 1)$ to get the estimate

$$\begin{aligned} \int_0^{r_{j+1}^2} \int_{B^{j+1}} u^{\kappa a} dx dt &= \int_0^{r_{j+1}^2} \int_{B^{j+1}} (u^{a/2} \zeta^{b/2})^{2\kappa} dx dt \\ &= \frac{C}{r_{j+1}^{n+2}} \iint_{U^{j+1}} (u^{a/2} \zeta^{b/2})^{2\kappa} dx dt \leq \frac{C2^{n+2}}{r_j^{n+2}} \iint_{U^j} (u^{a/2} \zeta^{b/2})^{2\kappa} dx dt \\ &\leq \frac{C2^{n+2}}{r_j^{n+2}} \iint_{U^j} |\nabla(u^{a/2} \zeta^{b/2})|^2 dx dt \left(\operatorname{ess\,sup}_{t \in (0, r_j^2)} \int_{B^j} (u^{a/2} \zeta^{b/2})^{n(\kappa-1)} dx \right)^{2/n}. \end{aligned}$$

In the previous inequality, we may bypass the boundedness assumption in the parabolic Sobolev's inequality by considering $\min\{u, k\}$ and using the monotone convergence theorem to pass to the limit. By the choice of a and b , we get the estimate

$$\begin{aligned} &|\nabla(u^{a/2} \zeta^{b/2})|^2 \\ &\leq C \left(\left(\frac{m - \varepsilon}{2} \right)^2 u^{m-\varepsilon-2} \zeta^{2(\frac{m-\varepsilon}{1-\varepsilon})} |\nabla u|^2 + \left(\frac{m - \varepsilon}{1 - \varepsilon} \right)^2 u^{m-\varepsilon} |\nabla \zeta|^2 \right) \\ &\leq C \left(u^{m-\varepsilon-2} \zeta^2 |\nabla u|^2 + \frac{1}{(1 - \varepsilon)^2} u^{m-\varepsilon} |\nabla \zeta|^2 \right). \end{aligned}$$

Here we used the fact that $\varepsilon \in (0, 1)$. Thus, by Lemma 2.4, we get

$$\begin{aligned} & \iint_{U^j} |\nabla(u^{a/2}\zeta^{b/2})|^2 dx dt \\ & \leq C \iint_{U^j} \left(u^{m-\varepsilon-2}\zeta^2 |\nabla u|^2 + \frac{1}{(1-\varepsilon)^2} u^{m-\varepsilon} |\nabla \zeta|^2 \right) dx dt \\ & \leq \frac{C}{|\varepsilon|^2(1-\varepsilon)^2} \left(\iint_{U^j} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \iint_{U^j} u^{1-\varepsilon} |\zeta_t| dx dt \right). \end{aligned}$$

Again, by the choice of a , b and κ , we have $u^{\frac{a}{2}n(\kappa-1)}\zeta^{\frac{b}{2}n(\kappa-1)} = u^{1-\varepsilon}\zeta^2$ and thus by Lemma 2.4, we get

$$\begin{aligned} & \left(\operatorname{ess\,sup}_{t \in (0, r_j^2)} \int_{B^j} (u^{a/2}\zeta^{b/2})^{n(\kappa-1)} dx \right)^{2/n} \\ & \leq \left(\frac{C}{|\varepsilon|^2(1-\varepsilon)^2} \left(\iint_{U^j} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \iint_{U^j} u^{1-\varepsilon} |\zeta_t| dx dt \right) \right)^{2/n}. \end{aligned}$$

So far we have

$$\begin{aligned} & \int_0^{r_{j+1}^2} \int_{B^{j+1}} u^{\kappa a} dx dt \leq \\ & \frac{C}{r_j^{n+2}} \left(\frac{1}{\varepsilon^2(1-\varepsilon)^2} \iint_{U^j} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \iint_{U^j} u^{1-\varepsilon} |\zeta_t| dx dt \right)^{1+2/n} \leq \\ & \left(\frac{C}{\varepsilon^2(1-\varepsilon)^2} \frac{2^{2(j+1)}}{(1-\alpha)^2} \left(\int_0^{r_j^2} \int_{B^j} u^{m-\varepsilon} dx dt + \int_0^{r_j^2} \int_{B^j} u^{1-\varepsilon} dx dt \right) \right)^{1+2/n} \end{aligned}$$

We denote $\sigma = \frac{1-\varepsilon}{m-\varepsilon}$. Now, Hölder's inequality gives us

$$\begin{aligned} & \int_0^{r_{j+1}^2} \int_{B^{j+1}} u^{m+\frac{2}{n}-(1+\frac{2}{n})\varepsilon} dx dt \leq \\ & \left(\frac{C}{\varepsilon^2(1-\varepsilon)^2} \frac{2^{2(j+1)}}{(1-\alpha)^2} \left(\int_0^{r_j^2} \int_{B^j} u^{m-\varepsilon} dx dt + \left(\int_0^{r_j^2} \int_{B^j} u^{m-\varepsilon} dx dt \right)^\sigma \right) \right)^{1+2/n}. \end{aligned} \tag{4.1}$$

Next, we denote $\gamma = 1 + \frac{2}{n}$ and $\varepsilon_0 = 1 - \gamma^{-(N+1)}(q - (m-1))$. Let $\varepsilon_j = 1 - \gamma^j(1 - \varepsilon_0)$ and $\delta_j = m - \varepsilon_j$. Now we have

$$\begin{aligned} \delta_{N+1} &= m - \varepsilon_{N+1} = m + \gamma^{N+1}(\gamma^{-(N+1)}(q - (m-1)) - 1) = q \quad \text{and} \\ \delta_0 &= m - \varepsilon_0 = m - 1 + \left(1 + \frac{2}{n}\right)^{-(N+1)} (q - (m-1)) = s. \end{aligned}$$

We observe

$$\begin{aligned} m + \frac{2}{n} - \gamma\varepsilon_j &= m + \frac{2}{n} + \gamma^{j+1}(1 - \varepsilon_0) - \gamma = m + \gamma^{j+1}(1 - \varepsilon_0) - 1 \\ &= m - \varepsilon_{j+1} = \delta_{j+1}. \end{aligned}$$

Thus, denoting

$$\Lambda_j = \int_0^{r_j^2} \int_{B^j} u^{\delta_j} dx dt,$$

(4.1) can be written as

$$\Lambda_{j+1} \leq \left(\frac{C2^{2j}}{\varepsilon_j^2(1 - \varepsilon_j)^2(1 - \alpha)^2} (\Lambda_j + \Lambda_j^\sigma) \right)^\gamma.$$

In order to estimate the term $\frac{1}{\varepsilon_j^2(1 - \varepsilon_j)^2}$, we want to show that ε_j is an decreasing sequence, and that $\varepsilon_N > 0$. Consider the difference

$$\varepsilon_{j+1} - \varepsilon_j = (1 - \varepsilon_0)\gamma^j(1 - \gamma) = (\varepsilon_0 - 1)\gamma^j \frac{2}{n}.$$

By assumption $q > (m - 1)$, implying that $\varepsilon_0 < 1$. Hence the sequence ε_j is decreasing. On the other hand

$$\varepsilon_N = 1 - \gamma^{-1}(q - (m - 1)) > 0,$$

because $q < m + \frac{2}{n} = m - 1 + \gamma$, by assumption. We now have the estimate

$$\frac{1}{\varepsilon_j^2(1 - \varepsilon_j)^2} \leq \frac{1}{\varepsilon_N^2(1 - \varepsilon_0)^2} = \frac{\gamma^2}{(m + \frac{2}{n} - q)^2(s - (m - 1))^2}.$$

We denote $\lambda = \frac{1}{(m + \frac{2}{n} - q)(s - (m - 1))}$. Thus

$$\Lambda_{j+1} \leq \left(\frac{C2^{2j}\lambda^2}{(1 - \alpha)^2} (\Lambda_j + \Lambda_j^\sigma) \right)^\gamma. \quad (4.2)$$

By assumption

$$\Lambda_0 = \int_0^{\rho^2} \int_{B(x_0, \rho)} u^s dx dt \leq \tilde{C}.$$

Thus, iterating (4.2) $N + 1$ times gives

$$\int_0^{(\alpha\rho)^2} \int_{B(x_0, \alpha\rho)} u^q dx dt = \Lambda_{N+1} \leq C.$$

□

Next, we will show, that the L^q -norms of these supersolutions are uniformly bounded.

Lemma 4.2. *Let u be a weak supersolution in Ω_{T_0} , such that*

$$|\{x \in B(x_0, \rho) : u(x, t) > k^{1+\frac{1}{d}}\}| \leq k^{-\frac{1}{d}}|B(x_0, \rho)|,$$

for every $k > 1$ and for almost every $t \in (0, \rho^2)$. Here d is as in Lemma 3.7. Then for $q \in (m-1, m + \frac{2}{n})$ we have

$$\int_0^{(\frac{3}{4}\rho)^2} \int_{B(x_0, \frac{3}{4}\rho)} u^q dx dt \leq C.$$

Proof. Let $\delta = \frac{1}{2d+2}$ and let $t \in (0, \rho^2)$. Then, by applying Cavalieri's principle at the time level t , we have

$$\begin{aligned} \int_{B(x_0, \rho)} u^\delta dx &= \delta \int_0^\infty \lambda^{\delta-1} |\{x \in B(x_0, \rho) : u(x, t) > \lambda\}| d\lambda \\ &= C \int_0^\infty k^{(1+\frac{1}{d})(\delta-1)} |\{x \in B(x_0, \rho) : u(x, t) > k^{1+\frac{1}{d}}\}| k^{\frac{1}{d}} dk \\ &\leq C \left(|B(x_0, \rho)| + \int_1^\infty k^{(1+\frac{1}{d})(\delta-1)} |\{x \in B(x_0, \rho) : u(x, t) > k^{1+\frac{1}{d}}\}| k^{\frac{1}{d}} dk \right) \\ &\leq C |B(x_0, \rho)| \left(1 + \int_1^\infty k^{-(1+\frac{1}{2d})} dk \right) \leq C |B(x_0, \rho)|. \end{aligned}$$

Here we used the assumption $|\{x \in B(x_0, \rho) : u(x, t) > k^{1+\frac{1}{d}}\}| k^{\frac{1}{d}} \leq |B(x_0, \rho)|$. Thus

$$\int_{B(x_0, \rho)} u^\delta dx \leq C \tag{4.3}$$

for almost every $t \in (0, \rho^2)$. Denote $U(s) = B(x_0, s) \times (0, s^2)$, for $s \in (\frac{7}{8}\rho, \rho)$. Let $\frac{7}{8}\rho \leq s < S \leq \rho$ and take a smooth cut-off function $\zeta \in C_0^\infty(B(0, S))$, such that $0 \leq \zeta \leq 1$ and

$$\begin{aligned} \zeta &= 1 \quad \text{in } U(s) \quad \text{and} \\ |\nabla \zeta| &\leq \frac{C}{S-s}. \end{aligned}$$

As in the proof of Lemma 4.1, we want to use the parabolic Sobolev's inequality and Caccioppoli estimates. We choose

$$\begin{aligned} a &= m-1 + \delta, \\ \kappa &= 1 + \frac{2\delta}{n(m-1+\delta)} \quad \text{and} \\ b &= 2 \end{aligned}$$

and thus we obtain

$$\begin{aligned} &\iint_{U(s)} u^{\kappa a} dx dt \\ &\leq \iint_{U(S)} |\nabla(u^{\frac{a}{2}} \zeta^{\frac{b}{2}})|^2 dx dt \left(\operatorname{ess\,sup}_{t \in (0, S^2)} \int_{B(x_0, S)} |u^{\frac{a}{2}} \zeta^{\frac{b}{2}}|^{(\kappa-1)n} dx \right)^{2/n}. \end{aligned}$$

We observe $\kappa a = m - 1 + \delta(1 + \frac{2}{n})$ and $|u^{\frac{a}{2}} \zeta^{\frac{b}{2}}|^{(\kappa-1)n} = u^\delta \zeta^{\frac{2\delta}{m-1+\delta}}$. Moreover, we may estimate

$$|\nabla(u^{\frac{a}{2}} \zeta^{\frac{b}{2}})|^2 \leq C(u^{m-3+\delta} \zeta^2 |\nabla u|^2 + u^{m-1+\delta} |\nabla \zeta|^2).$$

Combining these estimates, we have

$$\begin{aligned} \iint_{U(s)} u^{m-1+\delta(1+\frac{2}{n})} dx dt &\leq C \iint_{U(S)} \left(u^{m-3+\delta} \zeta^2 |\nabla u|^2 + u^{m-1+\delta} |\nabla \zeta|^2 \right) dx dt \\ &\quad \times \left(\operatorname{ess\,sup}_{t \in (0, S^2)} \int_{B(x_0, S)} u^\delta \zeta^{\frac{2\delta}{m-1+\delta}} dx \right)^{2/n}. \end{aligned}$$

Since $\zeta \leq 1$, we may use (4.3) to get

$$\operatorname{ess\,sup}_{t \in (0, S^2)} \int_{B(x_0, S)} u^\delta \zeta^{\frac{2\delta}{m-1+\delta}} dx \leq \operatorname{ess\,sup}_{t \in (0, \rho^2)} \int_{B(x_0, \rho)} u^\delta dx \leq C \rho^n.$$

We may set $\varepsilon = 1 - \delta$ and use Lemma 2.4 to get the estimate

$$\begin{aligned} &\iint_{U(S)} u^{m-\varepsilon-2} \zeta^2 |\nabla u|^2 dx dt + \iint_{U(S)} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt \\ &\leq C \iint_{U(S)} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \int_{B(x_0, S)} u^{1-\varepsilon}(x, S^2) \zeta(x)^2 dx \\ &\leq C \iint_{U(S)} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + C \rho^n. \end{aligned}$$

Here we used the fact that $\zeta_t = 0$ and (4.3). We have

$$\int_0^{s^2} \int_{B(x_0, s)} u^{m-1+\delta(1+\frac{2}{n})} dx dt \leq C \rho^2 \int_0^{S^2} \int_{B(x_0, S)} u^{m-1+\delta} |\nabla \zeta|^2 dx dt + C.$$

Next, we will use Young's inequality with

$$\begin{aligned} p &= \frac{m-1+\delta(1+\frac{2}{n})}{m-1+\delta} \quad \text{and} \\ q &= \frac{n(m-1+\delta(1+\frac{2}{n}))}{2\delta}. \end{aligned}$$

to get

$$\begin{aligned} &C \rho^2 \int_0^{S^2} \int_{B(x_0, S)} u^{m-1+\delta} |\nabla \zeta|^2 dx dt \\ &\leq C_1 \int_0^{S^2} \int_{B(x_0, S)} u^{(m-1+\delta)p} dx dt + C_2 \int_0^{S^2} \int_{B(x_0, S)} \rho^{2q} |\nabla \zeta|^{2q} dx dt \\ &\leq C_1 \int_0^{S^2} \int_{B(x_0, S)} u^{m-1+\delta(1+\frac{2}{n})} dx dt + C \left(\frac{\rho}{S-s} \right)^{n(m-1+\delta(1+2/n))/\delta}. \end{aligned}$$

We choose the constants in such a way, that $C_1 < 1$ and thus C is determined accordingly. We denote

$$\begin{aligned}\phi(s) &= \int_0^{s^2} \int_{B(x_0, s)} u^{m-1+\delta(1+\frac{2}{n})} dx dt \quad \text{and} \\ \sigma &= n(m-1+\delta(1+2/n))/\delta.\end{aligned}$$

Thus we have

$$\phi(s) \leq C\rho^\sigma (S-s)^{-\sigma} + C_1\phi(S).$$

Now by [7, Lemma 8.15]

$$\int_0^{s^2} \int_{B(x_0, s)} u^{m-1+\delta(1+\frac{2}{n})} dx dt \leq C \left(\frac{\rho}{S-s} \right)^\sigma.$$

Choosing $s = \frac{7}{8}\rho$ and $S = \rho$ shows that

$$\int_0^{(\frac{7}{8}\rho)^2} \int_{B(x_0, \frac{7}{8}\rho)} u^{m-1+\delta(1+\frac{2}{n})} dx dt \leq C.$$

Let $q \in (m-1, m+\frac{2}{n})$. Note, that the constant d in Lemma 3.7 can be chosen to be as large as we please. Thus we may choose d in such a way, that

$$\delta = \left(1 + \frac{2}{n}\right)^{-(N+1)} (q - (m-1)), \quad \text{for some } N.$$

Now, applying Lemma 4.1 in $B(x_0, \frac{7}{8}\rho) \times (0, (\frac{7}{8}\rho)^2)$, with $\alpha = \frac{6}{7}$ concludes the proof. \square

Next we will show the boundedness of the L^1 -norms of the gradients ∇u^m .

Lemma 4.3. *Let u be a weak supersolution in Ω_{T_0} , such that $u > 0$. Suppose, that*

$$|\{x \in B(x_0, \rho) : u(x, t) > k^{1+\frac{1}{d}}\}| \leq k^{-\frac{1}{d}} |B(x_0, \rho)|,$$

for every $k > 1$ and for almost every $t \in (0, \rho^2)$. Then

$$\int_0^{(\frac{5}{8}\rho)^2} \int_{B(x_0, \frac{5}{8}\rho)} |\nabla u^m| dx dt \leq \frac{C}{\rho}.$$

Proof. Denote $U^1 = B(x_0, \frac{5}{8}\rho) \times (0, (\frac{5}{8}\rho)^2)$ and $U^2 = B(x_0, \frac{7}{8}\rho) \times (0, (\frac{7}{8}\rho)^2)$.

Take $\varepsilon \in (0, \frac{2}{n})$ and use Hölder's inequality to get

$$\begin{aligned}\frac{1}{|U^1|} \iint_{U^1} |\nabla u^m| dx dt &= \frac{1}{|U^1|} \iint_{U^1} m u^{m-1} |\nabla u| dx dt \\ &= \frac{1}{|U^1|} \iint_{U^1} m u^{\frac{m-\varepsilon-2}{2}} |\nabla u| u^{\frac{m+\varepsilon}{2}} dx dt \\ &\leq \left(\frac{1}{|U^1|} \iint_{U^1} m u^{m-\varepsilon-2} |\nabla u|^2 dx dt \right)^{1/2} \left(\frac{1}{|U^1|} \iint_{U^1} u^{m+\varepsilon} dx dt \right)^{1/2}.\end{aligned}$$

Let ζ be a smooth cut-off function, such that $0 \leq \zeta \leq 1$ and

$$\begin{aligned}\zeta &= 1 \quad \text{in } U^1, \\ \zeta &= 0 \quad \text{on } \partial^p U^2, \\ |\nabla \zeta| &\leq \frac{C}{\rho} \quad \text{and} \\ |\zeta_t| &\leq \frac{C}{\rho^2}.\end{aligned}$$

We may use Lemma 2.4 to control the first term on the right hand side

$$\begin{aligned}& \frac{1}{|U^1|} \iint_{U^1} m u^{m-\varepsilon-2} |\nabla u|^2 dx dt \\ & \leq \frac{C}{\varepsilon^2(1-\varepsilon)} \left(\frac{1}{|U^2|} \iint_{U^2} u^{m-\varepsilon} |\nabla \zeta|^2 dx dt + \frac{1}{|U^2|} \iint_{U^2} u^{1-\varepsilon} \zeta |\zeta_t| dx dt \right) \\ & \leq \frac{C}{\varepsilon^2(1-\varepsilon)\rho^2} \left(\frac{1}{|U^2|} \iint_{U^2} u^{m-\varepsilon} dx dt + \frac{1}{|U^2|} \iint_{U^2} u^{1-\varepsilon} dx dt \right) \\ & \leq \frac{C}{\varepsilon^2(1-\varepsilon)\rho^2} \frac{1}{|U^2|} \iint_{U^2} u^{m-\varepsilon} dx dt \\ & + \frac{C}{\varepsilon^2(1-\varepsilon)\rho^2} \left(\frac{1}{|U^2|} \iint_{U^2} u^{m-\varepsilon} dx dt \right)^{(1-\varepsilon)/(m-1)}.\end{aligned}$$

We may assume, that ε is chosen in such a way, that we have $m-1 < m-\varepsilon < m+\varepsilon < m+\frac{2}{n}$ and thus we may use Lemma 4.2 to conclude

$$\begin{aligned}& \int_0^{(\frac{5}{8}\rho)^2} \int_{B(x_0, \frac{5}{8}\rho)} |\nabla u^m| dx dt \\ & \leq \frac{C}{\rho} \left(\frac{1}{|U^2|} \iint_{U^2} u^{m-\varepsilon} dx dt + \left(\frac{1}{|U^2|} \iint_{U^2} u^{m-\varepsilon} dx dt \right)^{(1-\varepsilon)/(m-1)} \right)^{1/2} \\ & \times \left(\frac{1}{|U^1|} \iint_{U^1} u^{m+\varepsilon} dx dt \right)^{1/2} \\ & \leq \frac{C}{\rho}.\end{aligned}$$

□

Now, we will show, that the previous lemma gives us a uniform lower bound for the integral averages of u for almost every time level $t \in \left(0, \left(\frac{5}{8}\rho\right)^2\right)$, if the integral average at $t = 0$ is large enough.

Lemma 4.4. *Let u be a non-negative weak supersolution in Ω_{T_0} , such that*

$$|\{x \in B(x_0, \rho) : u(x, t) > k^{1+\frac{1}{d}}\}| \leq k^{-\frac{1}{d}} |B(x_0, \rho)|,$$

for every $k > 1$ and for almost every $t \in (0, \rho^2)$. There exists $C > 0$, such that if

$$\int_{B(x_0, \frac{1}{2}\rho)} u(x, 0) dx \geq 2C,$$

then

$$\operatorname{ess\,inf}_{t \in (0, (\frac{5}{8}\rho)^2)} \int_{B(x_0, \frac{5}{8}\rho)} u(x, t) dx \geq C.$$

Proof. Let $\zeta \in C_0^\infty(B(x_0, \frac{5}{8}\rho))$ be a cut-off function, such that $0 \leq \zeta \leq 1$ and

$$\begin{aligned} \zeta &= 1 \quad \text{in } B\left(x_0, \frac{1}{2}\rho\right) \quad \text{and} \\ |\nabla \zeta| &\leq \frac{\tilde{C}}{\rho}. \end{aligned}$$

By (2.4) we have

$$\begin{aligned} &\int_{B(x_0, \frac{5}{8}\rho)} u(x, \tau) \zeta(x) dx \\ &\geq \int_{B(x_0, \frac{5}{8}\rho)} u(x, 0) \zeta(x) dx - \int_0^\tau \int_{B(x_0, \frac{5}{8}\rho)} |\nabla u^m \cdot \nabla \zeta| dx dt \end{aligned}$$

for almost every $\tau \in (0, (\frac{5}{8}\rho)^2)$. Using Lemma 4.3, we may estimate

$$\int_0^\tau \int_{B(x_0, \frac{5}{8}\rho)} |\nabla u^m \cdot \nabla \zeta| dx dt \leq \frac{\tilde{C}}{\rho} \int_0^\tau \int_{B(x_0, \frac{5}{8}\rho)} |\nabla u^m| dx dt \leq C.$$

Thus, we have

$$\begin{aligned} &\int_{B(x_0, \frac{5}{8}\rho)} u(x, \tau) \zeta(x) dx \\ &\geq \int_{B(x_0, \frac{1}{2}\rho)} u(x, 0) dx - \int_0^\tau \int_{B(x_0, \frac{5}{8}\rho)} |\nabla u^m \cdot \nabla \zeta| dx dt \\ &\geq C \end{aligned}$$

for almost every $\tau \in (0, (\frac{5}{8}\rho)^2)$. We conclude

$$\operatorname{ess\,inf}_{t \in (0, (\frac{5}{8}\rho)^2)} \int_{B(x_0, \frac{5}{8}\rho)} u(x, t) dx \geq C.$$

□

We will prove the following simple lemma for the readers convenience.

Lemma 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let f be a measurable function in Ω . Suppose that*

$$\int_{\Omega} f \, dx \geq 2C \quad \text{and} \quad \left(\int_{\Omega} f^q \, dx \right)^{1/q} \leq \lambda C,$$

for some $\lambda \geq 2$ and $q \in (1, \infty]$. Then

$$|\{x \in \Omega : f(x) > C\}| \geq \lambda^{\frac{-q}{q-1}} |\Omega|.$$

Proof. We have

$$\begin{aligned} 2C &\leq \int_{\Omega} f \, dx = \frac{1}{|\Omega|} \left(\int_{\{f>C\}} f \, dx + \int_{\{f \leq C\}} f \, dx \right) \\ &\leq \frac{1}{|\Omega|} \int_{\{f>C\}} f \, dx + C. \end{aligned}$$

Denote $\Omega_C = \{x \in \Omega : f(x) > C\}$. We use Hölder's inequality to control the first term on the right hand side by

$$\begin{aligned} \frac{1}{|\Omega|} \int_{\Omega_C} f \, dx &\leq \frac{1}{|\Omega|} \left(\int_{\Omega_C} f^q \, dx \right)^{1/q} |\Omega_C|^{(q-1)/q} \\ &\leq \left(\int_{\Omega} f^q \, dx \right)^{1/q} \left(\frac{|\Omega_C|}{|\Omega|} \right)^{(q-1)/q} \\ &\leq \lambda C \left(\frac{|\Omega_C|}{|\Omega|} \right)^{(q-1)/q}. \end{aligned}$$

Thus we have

$$2C \leq \lambda C \left(\frac{|\Omega_C|}{|\Omega|} \right)^{(q-1)/q} + C,$$

which implies

$$|\{x \in \Omega : f(x) > C\}| = |\Omega_C| \geq \lambda^{\frac{-q}{q-1}} |\Omega|,$$

concluding the proof. \square

Finally, we collect the results and apply Lemma 4.5 to show that we'll end up in a situation, where we can apply the expansion of positivity.

Lemma 4.6. *Let $d > 1$ be as in Lemma 3.7 and let u be a weak supersolution in Ω_{T_0} , such that*

$$|\{x \in B(x_0, \rho) : u(x, t) > k^{1+\frac{1}{d}}\}| \leq k^{-\frac{1}{d}} |B(x_0, \rho)|$$

for every $k > 1$ and for almost every $t \in (0, \rho^2)$. There exists $M > 0$, such that if

$$\int_{B(x_0, \frac{1}{2}\rho)} u(x, 0) \, dx \geq M,$$

then there exists $\tau \in \left(0, \left(\frac{5}{8}\rho\right)^2\right)$, such that

$$\left| \left\{ x \in B\left(x_0, \frac{5}{8}\rho\right) : u(x, \tau) > C_1 \right\} \right| \geq C_2 \left| B\left(x_0, \frac{5}{8}\rho\right) \right|,$$

for some $C_1, C_2 > 0$ depending only on m and n .

Proof. Let C be the constant given by Lemma 4.4. For M large enough, we have

$$\int_{B(x_0, \frac{1}{2}\rho)} u(x, 0) dx \geq 2C,$$

and thus we may use Lemma 4.4 to get

$$\operatorname{ess\,inf}_{t \in \left(0, \left(\frac{5}{8}\rho\right)^2\right)} \int_{B(x_0, \frac{5}{8}\rho)} u(x, t) dx \geq C.$$

Take $q > 1$. By Lemma 4.2, we have

$$\int_0^{\left(\frac{5}{8}\rho\right)^2} \int_{B(x_0, \frac{5}{8}\rho)} u^q dx dt \leq \tilde{C} \int_0^{\left(\frac{3}{4}\rho\right)^2} \int_{B(x_0, \frac{3}{4}\rho)} u^q dx dt \leq \tilde{C}.$$

We may choose $\tau \in \left(0, \left(\frac{5}{8}\rho\right)^2\right)$ and $\lambda \geq 2$, such that

$$\int_{B(x_0, \frac{5}{8}\rho)} u(x, \tau)^q dx \leq \lambda C.$$

Now, applying Lemma 4.5 gives

$$\left| \left\{ x \in B\left(x_0, \frac{5}{8}\rho\right) : u(x, \tau) > C \right\} \right| \geq \lambda^{\frac{-q}{q-1}} \left| B\left(x_0, \frac{5}{8}\rho\right) \right|.$$

Therefore, the Lemma holds for $C_1 = C$ and $C_2 = \lambda^{\frac{-q}{q-1}}$. \square

5. PROOF OF THEOREM 1.1

We are now ready to prove the main theorem. The idea of the proof is the following. Either u is large in a large portion of the ball $B(x_0, \rho)$ at some time level s or this does not happen at any time level. In the first case we may apply Lemma 3.7 to show, that u is essentially bounded from below by $\eta > 0$. In the latter case we utilize Lemma 4.6 to end up in a situation, where Lemma 3.7 can be applied. After this, the conclusion follows from a scaling argument.

Proof of Theorem 1.1. Denote

$$N = \int_{B(x_0, \rho)} u(x, t_0) dx.$$

We may assume $N > 0$. Define a scaled function

$$v(x, t) = \frac{M}{N} u\left(2x, t_0 + \left(\frac{M}{N}\right)^{m-1} t\right),$$

where M is as in Lemma 4.6. Since u is a weak supersolution in Ω_T , v is a weak supersolution in $\tilde{\Omega} \times \left(0, \left(\frac{N}{M}\right)^{m-1} (T_0 - t_0)\right)$. Moreover, v has the property

$$\int_{B(x_0, \frac{1}{2}\rho)} v(x, 0) dx = M.$$

One of the following alternatives holds. Either, there exists a time level $s \in (0, \rho^2)$ and a constant $k > 1$, such that

$$|\{x \in B(x_0, \rho) : v(x, s) > k^{1+\frac{1}{d}}\}| \geq \frac{1}{k^{\frac{1}{d}}} |B(x_0, \rho)| \quad (5.1)$$

or this does not hold for any pair s and k . In the spirit of [13], we call the former alternative “hot” and the latter “cold”. We will first consider the hot alternative. Suppose that there exists $s \in (0, \rho^2)$ and $k > 1$, such that (5.1) holds. Then, in particular, we have

$$|\{x \in B(x_0, \rho) : v(x, s) > k\}| \geq \frac{1}{k^{\frac{1}{d}}} |B(x_0, \rho)|$$

and thus Lemma 3.7 implies

$$v \geq \eta_0 \quad (5.2)$$

almost everywhere in $B(x_0, 2\rho)$ for almost every

$$t \in \left(\tilde{t} + \frac{1}{2} \frac{b^{m-1}}{\eta_0^{m-1}} \delta \rho^2, \tilde{t} + \frac{b^{m-1}}{\eta_0^{m-1}} \delta \rho^2\right),$$

where $\tilde{t} \in \left(s + \frac{1}{2} \frac{\delta}{k^{m-1}} \rho^2, s + \frac{\delta}{k^{m-1}} \rho^2\right)$. We approximate $\tilde{t} < 2\rho^2$. Since η_0 can be chosen to be as small as we please, we may assume, that (5.2) holds for almost every $t \in (2\rho^2 + \frac{1}{2}T_h, T_h)$, where $T_h = \frac{b^{m-1}}{\eta_0^{m-1}} \delta \rho^2$.

Next, we deal with the cold alternative. Suppose, that (5.1) does not hold for any pair s, k . Then, by Lemma 4.6, there exist $C_1, C_2 > 0$, such that

$$\left|\left\{x \in B\left(x_0, \frac{5}{8}\rho\right) : v(x, t) > C_1\right\}\right| \geq C_2 \left|B\left(x_0, \frac{5}{8}\rho\right)\right|.$$

Applying Lemma 3.7 twice and doing a similar approximation as in the hot alternative shows that there exist $\eta_1 > 0$ and $T_c = \frac{b^{m-1}}{\eta_1^{m-1}} \delta \rho^2$, such that

$$v \geq \eta_1 \quad (5.3)$$

almost everywhere in $B(x_0, 2\rho)$ for almost every $t \in (2\rho^2 + \frac{1}{2}T_c, T_c)$. Since either (5.2) or (5.3) holds, we may apply Lemma 3.7 once more (adjusting the constants as necessary) to find $\tilde{T} \geq \frac{\max\{T_h, T_c\}}{\rho^2}$ and $\nu > 0$, such that

$$v \geq \nu \quad \text{almost everywhere in } B(x_0, 2\rho) \times \left(\frac{1}{2}\tilde{T}\rho^2, \tilde{T}\rho^2\right)$$

if

$$\tilde{T}\rho^2 \leq \left(\frac{N}{M}\right)^{m-1} (T_0 - t_0). \quad (5.4)$$

For the function u , this reads

$$u \geq \frac{N}{M}\nu \quad \text{almost everywhere in } B(x_0, 4\rho)$$

for almost every

$$t \in \left(t_0 + \frac{1}{2} \left(\frac{M}{N} \right)^{m-1} \tilde{T}\rho^2, t_0 + \left(\frac{M}{N} \right)^{m-1} \tilde{T}\rho^2 \right).$$

Recalling the definition of N and choosing the constants $C_1 = \tilde{T}M^{m-1}$ and $C_2 = \frac{M}{\nu}$, we obtain

$$C_2 u \geq \int_{B(x_0, \rho)} u(x, t_0) dx \quad \text{almost everywhere in } B(x_0, 4\rho) \times \left(t_0 + \frac{1}{2}\tau, t_0 + \tau \right),$$

if (5.4) holds. Here

$$\tau = \min \left\{ T_0 - t_0, C_1 \rho^2 \left(\int_{B(x_0, \rho)} u(x, t_0) dx \right)^{1-m} \right\}.$$

If (5.4) does not hold, we have

$$\int_{B(x_0, \rho)} u(x, t_0) dx \leq \left(\frac{C_1 \rho^2}{T_0 - t_0} \right)^{1/(m-1)}$$

and so we conclude

$$\int_{B(x_0, \rho)} u(x, t_0) dx \leq \left(\frac{C_1 \rho^2}{T_0 - t_0} \right)^{1/(m-1)} + C_2 \operatorname{ess\,inf}_Q u,$$

where $Q = B(x_0, 4\rho) \times (t_0 + \frac{1}{2}\tau, t_0 + \tau)$, thus proving the theorem. □

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